ETH-TH/96-44

hep-th/9612178

From Path Representations to Global Morphisms for a Class of Minimal Models

by

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Abstract

We construct global observable algebras and global DHR morphisms for the Virasoro minimal models with central charge $c(2,q)$, q odd. To this end, we pass from the irreducible highest weight modules to path representations, which involve fusion graphs of the $c(2,q)$ models. The paths have an interpretation in terms of quasi-particles which capture some structure of non-conformal perturbations of the $c(2,q)$ models. The path algebras associated to the path spaces serve as algebras of bounded observables. Global morphisms which implement the superselection sectors are constructed using quantum symmetries: We argue that there is a canonical semi-simple quantum symmetry algebra for each quasi-rational CFT, in particular for the $c(2,q)$ models. These symmetry algebras act naturally on the path spaces, which allows to define a global field algebra and covariant multiplets therein.

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1. Introduction

The algebraic approach to quantum field theory as developed by Haag, Kastler, Doplicher and Roberts [37, 18] aims at a formulation of quantum theory in terms of observable quantities only: A theory is defined by a net of local observables, i.e. an assignment $I \longrightarrow \mathcal{A}(I)$, where $\mathcal{A}(I)$ is the *-algebra generated by all bounded "measurements" (observables) that can be made in some bounded region I of space-time. This net is subject to certain explicit axioms like isotony and causality (or rather Haag duality). Different superselection sectors of the quantum theory correspond to inequivalent positive energy representations of the local net on Hilbert spaces $\mathcal{H}_i$; the representations have to be covariant with respect to some group of space-time transformations (including the Poincaré group); among the representations, there is a vacuum sector $\mathcal{H}_0$. With the help of a localization criterion – which a priori rules out charges of gauge theories, but is tailor-made for short-range strong interactions as well as for charges occurring in $1+1$ dimensional conformal field theories – Doplicher, Haag and Roberts were able to determine a class of representations π_i which can be described by algebra endomorphisms ρ_i of the local net: $\pi_i \simeq \pi_0 \circ \rho_i$. From the observable algebra and the endomorphisms, one can reconstruct a field algebra containing charged operators.
The DHR formulation offers a good framework for abstract and mathematically precise investigations. With the endomorphisms, it provides a unique and efficient tool absent from any other approach to QFT. In particular, the DHR endomorphisms allow for a natural definition of "fusion" of representations simply as composition of endomorphisms, and they make it possible to study the statistical properties of superselection sectors in a quantitative fashion, culminating in the duality theorem of Doplicher and Roberts which completely classifies the fusion structures that are realized by local QFT in $3+1$ dimensions [19]. For precise statements and further details, see e.g. [36] and the references quoted there.
Although originally designed for the study of QFT in physical Minkowski space-time, the DHR framework can be applied to conformal field theory in $1+1$ dimensions as well, without the need to change the basic axioms [24]. Fredenhagen, Rehren and Schroer could show that algebraic QFT provides an entirely natural and computationally effective treatment of the phenomenon of braid group statistics [27] typical for low-dimensional quantum field theories, and again the DHR endomorphisms prove very effective for classifying the braid group representations realized by the sectors, see also [28, 52].
The DHR approach has led to a number of theorems on the general structure of chiral conformal field theories; here we mention only Longo's construction of Jones inclusions from endomorphisms [43, 55] and Wiesbrock's results [64] – based on earlier work by Borchers and others – on the realization of the Möbius group SU(1,1) through the Tomita-Takesaki modular operators, see also [33].

On the other hand, the "conventional approach" to CFT, initiated in the work [8], and in [26, 42, 65], has its merits, too. Compared to the DHR formulation, it is conceptually less stringent and less rigorous mathematically, but much more successful when it comes to the description of specific models. A theory is defined in terms of explicit generators (unbounded Laurent modes of point fields) and (commutation) relations, and representation theory of Lie algebras plays a prominent role. In contrast to the von Neumann algebra

approach of the algebraic formulation, unitary and non-unitary models can essentially be treated on the same footing. The study of models has led to a variety of interesting results which have no counterpart in the algebraic approach, among them the construction of W-algebras and the discovery of the role of the modular group, which poses severe restrictions onto the possible space of states. Beyond that, it has been possible to fit CFT into the wider frame of string theory and of two-dimensional integrable models, including models of statistical mechanics on the lattice.
If one had been limited to the investigation of general properties without the possibility to analyze specific models, many of these aspects would probably have gone unnoticed. Therefore, we think that it would be beneficial for algebraic QFT to try to discuss models in its language; in turn, the conceptual power of the DHR approach should provide valuable new insights e.g. for the complete classification of conformal field theories.

Model-building is rather problematical in the algebraic framework. Starting from scratch with an abstract net of von Neumann algebras and identifying the corresponding conformal model only from the relative positions of the local algebras, has, to our knowledge, never been carried out in concrete examples. Alternatively, one can take a model given in the "conventional" formulation and try to "translate" into the algebraic language. The "dictionary", see e.g. [30], is easily set up for some parts of the theory, e.g. the statistical structure, and even the construction of a local net is, formally, straightforward: One smears out the point fields with test functions supported in an interval $I \subset S^1$ and forms bounded functions of the resulting operators; these generate $\mathcal{A}(I)$. Covariance of the net is guaranteed if one starts with unbounded fields that are (quasi-)primary with respect to the central extension of $\mathrm{Diff}(S^1)$. However, in forming such local algebras, the advantages of working with concrete (unbounded) generators and relations are lost, and it is in general not possible to construct DHR morphisms explicitly.

A few conformal models can be treated in the DHR framework, at least to some extent. The simplest example is the "free boson", more precisely the CFT associated to the U(1) current algebra, which is discussed in great detail in [15]. There, the local algebras $\mathcal{A}(I)$ have (bounded) generators obeying Weyl type relations, which are simple enough to gain complete control of the local net, and one can also explicitly determine the localized DHR endomorphisms (which are automorphisms in this model). In the cases of non-abelian WZW models, loop group representation theory is a useful tool for the study of local nets [33, 63], but DHR endomorphisms have not been constructed along these lines.

A slightly different approach has been used in [44], where the Ising model was reformulated by algebraic methods. One can again introduce local observable algebras as above, but for the explicit construction of DHR endomorphisms Mack and Schomerus exploited the fact that there is a natural C^*-algebra associated to the Ising model: The maximally extended chiral symmetry algebra of the Ising model is generated by a free fermion, the Laurent modes b_r of which are bounded operators and form a global algebra of bounded observables. More precisely, observables are bilinear in the fermion modes, and the full algebra decomposes into the direct sum of Neveu-Schwarz and Ramond sector. Since the generators b_r satisfy simple anti-commutation relations, it is not too difficult to give formulas for global endomorphisms. In [44], it was shown that these implement the superselection sectors and fusion rules familiar from the Ising model; furthermore, a field algebra with

quantum symmetry action was constructed and the braid group representations associated with the superselection structure were identified. The authors also gave plausible arguments why their global endomorphisms are equivalent to localized ones, which was proven only later in [13] using more elaborate operator algebraic techniques.

The results of [44] were useful for a number of other models, too, since there exist free fermion representations of e.g. all level 1 WZW models [31]. The morphisms of these theories are very similar to those of the Ising model. With additional effort, using "multi-flavor" fermion representations, one can also describe some aspects of certain higher level WZW models in algebraic terms [14].

In the case of a conformal QFT, the von Neumann algebras forming a local net are all isomorphic to the hyperfinite factor of type III_1 and display no "individual structure". The global C^*-algebra of [44], on the other hand, does have some "model-characterizing structure" which, to some extent, substitutes for the explicit generators and relations of the conventional approach. If we want to find an algebraic description of other conformal theories like the minimal models of the Virasoro algebra, and in particular want to find concrete formulas for DHR morphisms, we think that it is most promising to follow this C^*-algebraic approach. However, in trying to do so, we meet difficulties at once, simply because, unlike the Ising model, the chiral algebras of the other Virasoro minimal models have no generating fields with bounded Laurent modes. Thus, we have to look for a new device to supply us with a C^*-algebra of bounded observables which can be used as a starting point.

Such a device is available for the special sub-series of minimal models with central charge $c(2,q)$, q odd: Instead of working with Virasoro modes directly, we first introduce *path representations* for the irreducible highest weight modules of the $c(2,q)$ models. These are vector spaces generated by (finite) paths on certain graphs. The graphs carry labels which make it possible to define an "energy grading" on the path spaces, and one can show that for each irreducible $c(2,q)$ module there is a labeled graph such that the spectrum of the path energy operator coincides with the true energy spectrum in the CFT highest weight representation. Thus, the most important, and the only invariant, information contained in the state spaces can be captured on a labeled graph. It turns out that one may use the same graph for all sectors of a given model and that only the path starting conditions depend on the sector. Incidentally, the relevant graph is the fusion graph of a certain primary field in the given $c(2,q)$ model.

There is evidence that the path representations themselves are physically relevant: They admit an interpretation in terms of quasi-particles which contains information on non-conformal deformations of the $c(2,q)$ models. But we can also view the construction of the path spaces as a mere representation theoretic interlude (of a somewhat unusual kind), and exploit it in order to obtain a natural global C^*-algebras of bounded observables that characterize our models: There is a canonical C^*-algebra acting on each path space, namely the *path algebra* over the respective graph. We take this algebra as the global observable algebra $\mathcal{A}$ in the corresponding representation, and identify the universal global observable algebra with the direct sum of these algebras over all sectors. As in the Ising case, $\mathcal{A}$ has non-trivial center, in agreement with the general theorems of [24]: For chiral

conformal QFTs, the center of $\mathcal{A}$ contains global charges (one per sector), and the vacuum representation is not faithful. Basically, this happens because on the "space-time" S^1 one cannot "hide an electron behind the moon".

There is another structural similarity between our $\mathcal{A}$ and the global Ising observable algebra, a more surprising one in view of the very different constructions: In both cases, the global algebras are *AF-algebras*, i.e. inductive limits of finite-dimensional matrix algebras. This property is obvious for the path algebras of the $c(2,q)$ models, whereas in the Ising model it can be seen after expressing bi-linears in the fermion modes through Temperley-Lieb-Jones generators, see [44].
Moreover, the algebras associated to different sectors of a given model – Ising or one of the $c(2,q)$ series – are mutually "stably isomorphic", i.e. they are isomorphic after tensoring with the compact operators. Thus, it appears that the stable isomorphism type of the global algebras characterizes the conformal model.

We believe that this is in fact a general phenomenon, and that to each rational conformal model there is an AF-algebra playing the role of the global observable algebra. Although we have no proof for this conjecture, there is some amount of evidence: Already within the algebraic approach, one is naturally led to study the AF-sub-algebras of the observable algebra which are generated by intertwiners [24, 28]. These contain all information on the statistics of the model, and in view of the very restrictive conformal covariance laws they determine at least a great deal of the global observable algebra of a CFT. Other examples of AF-algebras are provided by various kinds of lattice approximations of conformal models. The two-dimensional RSOS lattice models, see e.g. [6, 50], are by definition based on path spaces (in a sense different from ours). Similarly, the lattice current algebras introduced by Faddeev, which are one-dimensional discretizations of conformal models, display an AF-structure [2], which is somewhat more subtle but should become important precisely for studying the continuum limit of these lattice models. Since one expects to "obtain" conformal field theories in this limit, both from RSOS models and from lattice current algebras, it is natural to conjecture that AF-algebras show up in the continuum theory as well. One might hope that eventually conformal field theories can be classified by *invariants* of "their associated" AF-algebras (plus some extra data) similar to the classification of Lie algebras by Dynkin diagrams.

Coming back to the comparison of the Ising with the $c(2,q)$ models, we observe that in our cases it is much more difficult to define local sub-algebras of the global observable algebra $\mathcal{A}$ explicitly. In the Ising model, the fermion modes can be used both to generate the global observable algebra and to form a covariant local net. In our cases, one could build local algebras from the (smeared) energy momentum field, but these will be sub-algebras of the global path observable algebra only if we start from a Virasoro action on the paths. As we will see, the construction of the path representations "imports" a seemingly natural operation of the Virasoro algebra into the path spaces, but this action is very irregular in terms of paths and virtually impossible to control. One need not stick to this "natural" Virasoro action and can instead look for another implementation of Diff(S^1) or SU(1,1) which takes the specific structure of the path spaces into account and is, accordingly, easier to handle within the path algebra. Progress towards such representations of SU(1,1) has

been made in [56], but the results are not yet explicit enough to construct covariant local nets. In view of this, we have to be content with an algebraic treatment of the global aspects of the $c(2,q)$ minimal models at present.

The next aim is to construct (global) *morphisms* of the (global) path observable algebra which implement the superselection sectors. Again, we cannot simply copy the procedure of [44] used for the Ising model, where it was the existence of the free fermion modes which allowed to "guess" the right endomorphisms. In the work [62], however, Vecserneys gave an alternative construction of morphisms for the Ising model, starting from an action of a semi-simple quantum symmetry algebra (QSA) on the Hilbert space of states. In this context, the morphisms necessarily arise in the form of (non-unital) *amplimorphisms* of the observable algebra – i.e. as maps from $\mathcal{A}$ into some matrix amplification $M_n(\mathcal{A})$ – and not as (unital) endomorphisms. The representations they implement are, however, equivalent to the ones obtained in [44].

Here, we will follow the procedure of [62] and use an action of a suitably chosen quantum symmetry algebra on the path spaces associated to a $c(2,q)$ model. For our purposes, we will not need the full structure of a QSA but only the information encoded in a co-product. The action of the QSA on the $c(2,q)$ path spaces leads to a natural global path field algebra $\mathcal{F}$ and to equations defining covariant field multiplets, which are elements in amplifications of $\mathcal{F}$. The charged multiplets immediately yield the desired amplimorphisms.

The paper is organized as follows: In the next section, we list the basic data of the $c(2,q)$ minimal models and then show how to construct path representations of the irreducible modules and of the global observable algebra. In addition, we present a physical interpretation of the path spaces in terms of quasi-particles. Section 3 starts with a few general remarks on quantum symmetries of low-dimensional QFTs; returning to our models, we make a special choice for the quantum symmetry algebra for the $c(2,q)$ theories and implement it on the path spaces – which leads to a natural definition of the global field algebra. In section 4, we first construct amplimorphisms of the QSA, which are used to formulate equations for covariant multiplets within the field algebra; finding solutions to these equations, which is achieved by a careful analysis of the combinatorial structure of the path spaces, is the main subject of that section. The then straightforward definition of global amplimorphisms of the path observable algebra is given in section 5, followed by a short discussion of properties of these maps. We conclude with an outlook on future developments. Subsections 2.3 on quasi-particles and also 3.2, where we justify the special (canonical) choice of our QSA, can be regarded as digressions which are not essential for the construction of amplimorphisms, but in our view they contain some considerations which are interesting in their own rights.

2. Path representations

We first set up notations and collect some facts about the $c(2,q)$ minimal models which will be used as input data for the following constructions. In subsection 2.2, we start with a brief review of some results of [22], where explicit bases for the irreducible modules of the $c(2,q)$ models were obtained. From there, it is relatively easy to give a re-interpretation of the FNO bases in terms of paths on certain graphs [40]. The resulting path representations probably bear a deeper physical meaning, which will be commented on in section 2.3. The immediate consequence relevant for our purposes is that they provide a partial operator algebraic description of the $c(2,q)$ conformal models which makes at least some of the DHR concepts applicable.

2.1 Data of the $c(2,q)$ minimal models

The conformal models we are going to study possess a maximal chiral observable algebra (the "energy-momentum W-algebra") generated by the energy momentum tensor

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \ ,$$

whose Laurent modes satisfy the Virasoro algebra

$$[\, L_n, L_m \,] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0} \tag{2.1}$$

with *central charge* c taken from the set

$$c(2,q) = -\frac{(3q-4)(q-3)}{q} \ , \quad q = 2K+3 \ , \quad K = 1,2,\dots \ . \tag{2.2}$$

The values (2.2) belong to the so-called minimal series of the Virasoro algebra [21, 9], and it is well known that for each of these central charges there exists a finite number of irreducible highest weight representations $\mathcal{H}_h$ with highest weights h in the Kac table such that those sectors together form a rational CFT. In our cases, the Kac table consists of the $K+1$ different *highest weights*

$$h_i = -\frac{i(q-2-i)}{2q} \ , \quad i = 0,\dots,K \ . \tag{2.3}$$

The distinguished feature of the charges and weights of the minimal models is the presence of two infinite chains of singular vectors in the associated Verma modules [21], and this property will be the background for the construction of path representations in the next subsection.

Among the minimal models, the sub-series (2.2,3) may be characterized as consisting of all the theories in which the energy-momentum W-algebra does not admit an extension

by additional bosonic or (para-)fermionic Virasoro primary fields. More precisely, they do not contain abelian sectors or "simple currents" [57]. This can be seen from the *fusion rules* of the $c(2,q)$ minimal models, which we will simply borrow from [8] and take as part of our input data; nevertheless, we will describe a procedure how to calculate the fusion rules in section 3.2.

Let us for the moment denote the irreducible sectors of the $c(2,q)$ minimal model, $q = 2K+3$, by the symbols ϕ_i, and composition (fusion) of two such sectors by $\phi_i \times \phi_j$, $i,j \in I_q := \{0,\ldots,K\}$. Then the representation of the observable algebra associated to $\phi_i \times \phi_j$ decomposes into irreducibles $\phi_i \times \phi_j = \sum_k N_{ij}^k \, \phi_k$ as

$$\phi_i \times \phi_j = \sum_{l=0}^{\min(i,j)} \phi_{\{|i-j|+2l\}} \tag{2.4}$$

where the abbreviation

$$\{m\} := \begin{cases} m & \text{if } 0 \leq m \leq K \\ 2K+1-m & \text{if } m > K \end{cases} \tag{2.5}$$

was used. Note that the number of terms on the rhs of (2.4) is

$$m_{ij} := \min\,(i+1, j+1)\ . \tag{2.6}$$

The fusion rules (2.4) can be regarded as kind of "orbifold" of the multiplication table of SU(2) representations, with the sector ϕ_1 playing the role of the fundamental representation: It obeys

$$\begin{aligned} &\phi_1 \times \phi_0 = \phi_1\ , \\ &\phi_1 \times \phi_i\ = \phi_{i-1} + \phi_{i+1} \quad \text{for} \quad 0 < i < K\ , \\ &\phi_1 \times \phi_K = \phi_{K-1} + \phi_K\ . \end{aligned}$$

Thus, its *fusion graph* – the graph with one node i per sector i and N_{1i}^j lines from node i to node j – has a tadpole form, with the only loop attached to node K. This graph arises from the Dynkin diagram A_{2K} (also familiar as the fusion graph of the fundamental representation sector in SU(2) WZW models) by dividing out the $\mathbb{Z}_2$-symmetry.

The abelian ring generated by the representations ϕ_i, or equivalently by the fusion matrices N_i with

$$\left(N_i\right)_{jk} := N_{ij}^k\ ,$$

is a *polynomial fusion ring* in the sense of [17]: It is spanned by ϕ_1 and $\check{p}_i(\phi_1)$, where $\check{p}_i(x)$ is the i th Čebyshev polynomial, see also [40].

Other distinguished sectors of the $c(2,q)$ models are the vacuum sector ϕ_0 with fusion rules $\phi_0 \times \phi_i = \phi_i$, and the sector ϕ_K corresponding to the representation with the *minimal conformal dimension* of all the values listed in (2.3), namely $h_K = -\frac{K(K+1)}{2(2K+3)}$. The fusion graph of ϕ_K, denoted $\mathcal{G}_q$, is described by the connectivity matrix

$$C_q = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 1 \\ \vdots & & \iddots & & \vdots \\ 0 & 1 & \cdots & & 1 \\ 1 & 1 & \cdots & & 1 \end{pmatrix} \tag{2.7}$$

which is simply the symmetric $(K+1) \times (K+1)$ fusion matrix of ϕ_K:

$$\left(C_q\right)_{ij} = \left(N_K\right)_{ij} \ .$$

The graphs $\mathcal{G}_q$, see Figure 1 for examples, will play a surprising role in the following.

2.2 Path representations and global observable algebra

Consider some highest weight representation $\mathcal{H}_h$ of the Virasoro algebra at central charge c with highest weight vector $|h\rangle$. It is well known that there exists a surjective homomorphism of Virasoro representations from the Verma module $\mathcal{V}_h$ (with the same highest weight h and central charge c) onto $\mathcal{H}_h$. Therefore, the images of the Poincaré-Birkhoff-Witt (PBW) vectors (which are a natural basis of $\mathcal{V}_h$)

$$L_{m_1} \dots L_{m_n} |h\rangle \tag{2.8}$$

where $m_1 \le \dots \le m_n < 0$ and $n \ge 0$, form a *spanning system* of $\mathcal{H}_h$. If $\mathcal{H}_h$ is an irreducible representation occurring in the minimal models, it contains null-vectors and the set (2.8) is linearly dependent. From the Kac determinant [21], the energy levels of the null-states are known, and there are also recursive formulas [9, 7] which in principle allow to compute them explicitly. Following a different route, Feigin, Nakanishi and Ooguri succeeded in singling out an linearly independent subset of (2.8) for c and h as in (2.2,3). A closer look at the construction given in [22] shows that the main fact to be exploited is simply that the vacuum vector is cyclic and separating for the local observable algebras. This implies that the non-trivial null-vectors of the vacuum representation $h = 0$ at $c = c(2, q)$ re-appear as local (point-like) null-fields which generate the "*annihilating ideal*" [22] of the energy-momentum W-algebra. The annihilating ideal is represented (by zero) in all the other sectors as well, which produces null-vectors there.

In the vacuum Verma module $\mathcal{V}_0$ with central charge $c(2, q)$, $q = 2K + 3$, there is the trivial singular vector $L_{-1}|0\rangle$ and a second one $|s^{(q)}\rangle$ at energy $q - 1$, see [21], which can be written as

$$|s^{(q)}\rangle = \lim_{z \to 0} N^{(q)}(z)|0\rangle = N^{(q)}_{1-q}|0\rangle$$

for a certain primary null-field $N^{(q)}(z)$ of conformal dimension $q - 1$. Using the notations of [47, 12], we may give a recursive formula for the normal ordered products $N^{(q)}(z)$

$$N^{(5)} = \mathcal{N}(T, T) \ , \qquad N^{(q)} = \mathcal{N}(T, N^{(q-2)}) \ , \tag{2.9}$$

see also [22]; the normal ordering prescription denoted by $\mathcal{N}(\cdot, \cdot)$ involves usual normal ordering and additional corrections which render the resulting field quasi-primary. For our purposes, the precise expression for the Laurent modes of $N^{(q)}(z)$ will be inessential, we only need the familiar part

$$N^{(q)}_n = \sum_{\substack{m_1, \dots, m_{K+1} \\ m_1 + \dots + m_{K+1} = n}} :L_{m_1} \dots L_{m_{K+1}}: \ + \ \dots \tag{2.10}$$

where the dots indicate the correction terms – which are polynomials in the L_m of *degree at most K.*

From the vacuum Verma module, we can pass to the irreducible vacuum representation by quotienting out the (intersecting) sub-Verma modules built over the singular vectors $|s^{(q)}\rangle$ and $L_{-1}|0\rangle$. Translating into the space of fields, this means that we have to set equal zero all descendants of the null-field $N^{(q)}(z)$, i.e. the whole annihilating ideal. (The equation $L_{-1}|0\rangle = 0$ translates into $\partial_z \mathbf{1} = 0$ and gives no information.) This means, in particular, that

$$N_n^{(q)}|v\rangle = 0 \tag{2.11}$$

for any mode of the null-field acting on any vector $|v\rangle$ in any of the representations $\mathcal{H}_i$ with highest weight h_i (2.3). Taking $|v\rangle$ to be one of the highest weight vectors $|h_i\rangle$, each of the eqs. (2.11) leads to linear relations among the PBW spanning system (2.8). Moreover, since $N^{(q)}(z)$ is primary and generates the full annihilating ideal, it is also clear that all linear dependencies in $\mathcal{H}_i$ arise from (2.11), or from equations $\phi_m N_n^{(q)}|v\rangle = 0$ with some other field $\phi(z)$ in the energy-momentum W-algebra. We can, therefore, determine a linearly independent subset of (2.8) from the null-field modes – provided we do the book-keeping correctly. This combinatorial problem can be solved by introducing a suitable *lexicographic ordering* of the PBW vectors: We refer to [22] for the details and mention only that one of the ordering criteria is the length k of the monomial $L_{m_1} \dots L_{m_k}$; this should make it plausible that formula (2.10) for $N_n^{(q)}$ is accurate enough for our purposes. Taking everything together, Feigin, Nakanishi and Ooguri prove at the following statement:

Proposition 2.1 [22] Let $\mathcal{H}_i$ be an irreducible highest weight representation of the $c(2, q)$ minimal model, $q = 2K + 3$, with highest weight h_i as in (2.3), and let $|h_i\rangle$ denote the highest weight vector. Then $\mathcal{H}_i$ has a basis consisting of those of the vectors

$$L_{m_1} \dots L_{m_n}|h_i\rangle$$

with $m_1 \leq \dots \leq m_n < 0$ and $n \geq 0$ that satisfy the "*difference two condition*"

$$m_{l+K} - m_l \geq 2 \tag{2.12}$$

for $1 \leq l \leq n - K$ as well as the "*initial condition*"

$$\sharp\{m_l = 1\} \leq i\ . \tag{2.13}$$

The difference two condition (2.12) controls whether monomials in the Virasoro generators are "too densely packed": All PBW vectors that contain (sub-)arrays also appearing in the expansion (2.10) of null-field modes are removed from the basis. Whereas (2.12) characterizes the $c(2, q)$ model as a whole, the initial condition (2.13), stating that the mode L_{-1} appears at most i times in the independent PBW monomials belonging to $\mathcal{H}_i$, allows to distinguish the irreducible representations from each other.

Let us now derive path representations from the FNO bases. We first notice that the PBW vectors (2.8), with h being one of the h_i in the list (2.3), can be uniquely expressed as *sequences* of positive integers via the identification

$$L_{-M}^{a_M} \dots L_{-2}^{a_2} L_{-1}^{a_1} |h_i\rangle \longmapsto (a_0(i); a_1, a_2, \dots, a_M, 0, \dots) \; ; \tag{2.14}$$

compared to (2.8), we have rewritten multiple equal modes as powers $L_{-m}^{a_m}$ and have also filled in the gaps (i.e. $a_m = 0$ is possible). The "dummy variable" $a_0(i)$ is used to number the different sectors h_i. After this reformulation, the difference two condition (2.12) takes the simple form

$$a_m + a_{m+1} \leq K \quad \text{for all } m > 0 \; , \tag{2.15}$$

which implies that

$$0 \leq a_m \leq K \; ; \tag{2.16}$$

the initial condition (2.13) of course means that

$$a_1 \leq i \; . \tag{2.17}$$

We can get rid of this extra inequality by choosing the numbering a_0 of sectors appropriately: Set

$$a_0(i) = K - i$$

in eq. (2.14), then the initial condition is nothing but the difference two condition (2.15) extended to $m = 0$.

We have accomplished a one-to-one map from the FNO basis vectors, Proposition 2.1, to sequences $(a_m)_{m\geq 0}$ of integers $a_m \in \{0, \dots, K\}$ which are subject to condition (2.16) for all $m \geq 0$. In the next step, we encode these restrictions into the following *labeled graph*: It consists of $K+1$ nodes which we denote by i running from 0 to K. The node i carries the label

$$l(i) := K - i \; . \tag{2.18}$$

The graph contains (precisely) one edge from node i to node j if the associated labels satisfy

$$l(i) + l(j) \leq K \; , \tag{2.19}$$

and none otherwise. This means that there are $i+1$ unoriented edges emanating from the node i. As it happens, the graph we have described is just the *fusion graph* $\mathcal{G}_q$ of the minimal dimension field ϕ_K, with its connectivity matrix given in eq. (2.7).

Now consider a *path* on $\mathcal{G}_q$, i.e. a sequence $(i_m)_{m\geq 0}$ where each i_m is a node of $\mathcal{G}_q$ such that there is an edge in $\mathcal{G}_q$ from the node i_m to the node i_{m+1} for all $m \geq 0$. Since the labeling (2.18) is unambiguous, each path of $\mathcal{G}_q$ is in one-to-one correspondence to a sequence of labels $(l(i_m))_{m\geq 0}$ which satisfy $0 \leq l(i_m) \leq K$ and $l(i_m) + l(i_{m+1}) \leq K$ – and these are precisely the requirements (2.15,16) on the sequences of a_m discussed above. Therefore, we have established a connection from the FNO bases of the irreducible modules of the $c(2,q)$ model to paths over the fusion graph $\mathcal{G}_q$.

Since the monomials occurring in the PBW system (2.8) are finite, we also have to restrict

to finite paths (to finite label sequences) – or to paths stabilizing at the node K, i.e. paths such that there exists an $M \gg 0$ with

$$i_m = K \quad \text{for all } m > M \tag{2.20}$$

($l(i_m) = 0$ for all $m > M$). In the following we will often speak of *finite paths* (of arbitrary length) when we mean infinite paths subject to the "tail condition" (2.20).
The initial condition (2.17), on the other hand, really becomes a simple initial condition for paths: The FNO basis vectors of the irreducible representation $\mathcal{H}_j$ correspond to the (finite) paths starting from node j (i.e. $i_0 = j$). We denote the complex vector space generated by those paths by $\mathcal{P}_j$.

Finally, our construction allows us to introduce an action of the energy operator L_0 on the path spaces $\mathcal{P}_i$. For each path $|p\rangle = (i_m)_{m\geq 0} \in \mathcal{P}_i$, let $(l_m)_{m\geq 0}$ be the associated sequence of labels. (Note that the associated sequences exist only for the paths themselves, not for arbitrary linear combinations.) We define an "energy operator" $L_0^{\mathcal{G}}$ on $\mathcal{P}_i$ by declaring each path to be an eigenvector of $L_0^{\mathcal{G}}$:

$$L_0^{\mathcal{G}}\,|p\rangle = \Big(h_i + \sum_{m\geq 0} m\, l_m\Big) \cdot |p\rangle \ ; \tag{2.21}$$

the constant h_i is purely conventional, but with this choice we recover precisely the energy values in the $c(2,q)$ representations. We summarize our results in the following statement:

Proposition 2.2 Let $\mathcal{G}_q$ be the (labeled) fusion graph of the minimal dimension field ϕ_K in the $c(2,q)$ model, $q = 2K+3$, defined by the connectivity matrix (2.7) and with labels as in eq. (2.18). For $i = 0, \ldots, K$, denote by $\mathcal{P}_i$ the vector space generated by all finite paths on $\mathcal{G}_q$ which start at node i. Then $\mathcal{P}_i$ and the irreducible highest weight representation $\mathcal{H}_i$ of the $c(2,q)$ model are isomorphic as $\mathbb{Z}$-graded spaces,

$$\mathcal{P}_i \cong_{L_0} \mathcal{H}_i \ ,$$

where the $\mathbb{Z}$-gradings are given by the usual energy operator L_0 on $\mathcal{H}_i$ and by the $L_0^{\mathcal{G}}$ action (2.21) on $\mathcal{P}_i$.

Let us emphasize that the path representations discussed here are not be confused with the path spaces underlying certain RSOS models, see e.g. [6, 50]. In particular, those paths have infinite tails that contribute non-trivially to the energy and even to the sector identification. They also do not admit the physical interpretation in terms of quasi-particles which will be discussed below.

Up to now, we have worked with unbounded Laurent modes of point-like fields. If we wish, we can view the previous constructions as a discussion of Virasoro algebra representation theory and now forget all the details. Then, we define the path spaces $\mathcal{P}_i$ over the graphs $\mathcal{G}_q$ as the state spaces of the $c(2,q)$ model. Together with the energy operator $L_0^{\mathcal{G}}$, which governs the "time evolution" of the models, they contain all the *invariant information* that can be extracted from the FNO basis.

In particular, we no longer need to identify "elementary" paths over $\mathcal{G}_q$ (i.e. true paths as opposed to linear combinations thereof) with states in the PBW system. The apparent advantage of this identification is that it would provide a concrete action of the Virasoro generators L_n on the path spaces. This "PBW type action", however, can hardly be cast into a closed form and bears no relation to the path structure. In order to obtain useful formulas for the Virasoro action, one has to apply more refined methods [56] based on a novel interpretation of paths, which will be sketched in section 2.3 below.

From the point of view of algebraic QFT, the unbounded Virasoro generators are not of prime importance, anyway. Instead, one would like to construct a covariant net of local observables $\big(\mathcal{A}(I),\ I\subset S^1\big)$. As was mentioned in the introduction, we are not able to give a local description of the $c(2,q)$ models, yet, but the path picture for the irreducible modules at least provides a natural algebra of bounded operators which we can view as the *global observable algebra*, more precisely the "universal" [24] observable algebra $\mathcal{A}\equiv\mathcal{A}_{\rm univ}$ generated by the $\mathcal{A}(I)$: Let $\mathcal{A}_i=\pi_i(\mathcal{A})$ denote the global (universal) observable algebra acting in the representation space $\mathcal{H}_i$ (or $\mathcal{P}_i$). With the elements of $\mathcal{A}_i$, we can map any state in $\mathcal{H}_i$ (or path in $\mathcal{P}_i$) into any other, therefore we take $\mathcal{A}_i$ to be the *path algebra* associated with $\mathcal{P}_i$.

To give a precise definition, we introduce some more notation. Let $\mathcal{P}^{(n)}_{i,j}$ be the space of paths of length n over $\mathcal{G}_q$ which run from node i to node j, i.e.

$$\mathcal{P}^{(n)}_{i,j}=\big\{\,|p\rangle=(i_0,\ldots,i_n)\,\big|\,i_m \text{ joined with } i_{m+1} \text{ on } \mathcal{G}_q,\ i_0=i,\ i_n=j\,\big\}_{\mathbb{C}}\ . \tag{2.22}$$

The finite-dimensional algebra $\mathcal{A}^{(n)}_{i,j}$ is linearly generated by so-called *strings* $|p\rangle\langle q|$, pairs of elementary paths $|p\rangle,|q\rangle\in\mathcal{P}^{(n)}_{i,j}$ which are multiplied like matrix units,

$$|p\rangle\langle q|\cdot|\tilde p\rangle\langle\tilde q|=\delta_{q,\tilde p}\,|p\rangle\langle\tilde q|$$

where $\delta_{q,\tilde p}$ is the Kronecker symbol. Strings act on paths in an analogous way, therefore $\mathcal{A}^{(n)}_{i,j}=\mathrm{End}\,(\mathcal{P}^{(n)}_{i,j})$. We can endow $\mathcal{P}^{(n)}_{i,j}$ with a scalar product making (elementary) paths pairwise orthonormal; then $\mathcal{A}^{(n)}_{i,j}$ becomes a *-algebra such that $\big(|p\rangle\langle q|\big)^*=|q\rangle\langle p|$. Denote by $\mathcal{A}^{(n)}_i$ the multi-matrix algebra $\mathcal{A}^{(n)}_i:=\bigoplus_j\mathcal{A}^{(n)}_{i,j}$.

Note that this standard scalar product on the path spaces does not directly reproduce the Shapovalov form on the irreducible modules via the original identification of PBW vectors with paths. But, as explained, this identification involved certain choices with no invariant meaning. Since the (invariant) L_0 eigenspaces are finite-dimensional, it is in principle possible to define a new (pseudo) scalar product on the spaces of fixed energy so as to reproduce the Shapovalov form, but a closed formula for all energy levels is not known. In this context, the quasi-particle interpretation of paths seems to be useful, see [56] for further comments.

If the graph $\mathcal{G}_q$ contains an edge from the node j to the node k, we can embed the path space $\mathcal{P}^{(n)}_{i,j}$ into $\mathcal{P}^{(n+1)}_{i,k}$ by *concatenation* of this edge to elementary paths; we denote the corresponding linear map by

$$c^j_k\ :\ \mathcal{P}^{(n)}_{i,j}\longrightarrow\mathcal{P}^{(n+1)}_{i,k}\ ; \tag{2.23}$$

note that c_k^j is independent of the starting node i. Concatenation induces an embedding of the simple factor $\mathcal{A}_{i,j}^{(n)}$ of $\mathcal{A}_i^{(n)}$ into $\mathcal{A}_{i,k}^{(n+1)}$ whenever j and k are joined by an edge of $\mathcal{G}_q$, i.e. whenever $\left(C_q\right)_{jk} = 1$, see (2.7). In other words, C_q is the *embedding matrix* of the *Bratteli diagram* associated to the injection

$$\Phi^{(n)} \,:\, \mathcal{A}_i^{(n)} \longrightarrow \mathcal{A}_i^{(n+1)} \tag{2.24}$$

induced by (2.23). In turn, this diagram determines $\Phi^{(n)}$ up to inner isomorphism in $\mathcal{A}_i^{(n+1)}$, see e.g. [11, 35].

Definition 2.3 The global path observable algebra of the $c(2,q)$ minimal model is $\mathcal{A} = \mathcal{A}_0 \oplus \ldots \oplus \mathcal{A}_K$ where $\mathcal{A}_i$ is the C^*-closure of the inductive limit of the system $\left(\mathcal{A}_i^{(n)}, \Phi^{(n)}\right)$ with embeddings (2.24) induced by concatenation of paths.

By construction, each generator L_n of the Virasoro algebra, restricted to finite energy subspaces of the representation $\mathcal{H}_i$, can be expressed by elements of $\mathcal{A}_i$; in particular, the "conformal Hamiltonian" $L_0^{\mathcal{G}}$ of (2.21) is "affiliated" to $\mathcal{A}$ in the sense that all its spectral projections are contained in $\mathcal{A}$. Instead of $\mathcal{A}_i$, we will sometimes write $\mathcal{A}_i = \pi_i(\mathcal{A}) \equiv \mathrm{pr}_i(\mathcal{A})$.

Each $\mathcal{A}_i$ is an *AF-algebra*, i.e. an "approximately finite-dimensional" algebra, see e.g. [11, 35]. The most convenient way to picture AF-algebras (up to isomorphism) is by the (infinite) Bratteli diagram associated to the "tower" $\left(\mathcal{A}_i^{(n)}, \Phi^{(n)}\right)$. This is an infinite graph, subdivided into *floors* which correspond to the finite-dimensional sub-algebras $\mathcal{A}_i^{(n)}$, $n = 0, 1, \ldots$. The nth floor consists of as many nodes as $\mathcal{A}_i^{(n)}$ has simple factors (in our case: one node on the zeroth, $K+1$ nodes on any other floor), and the nodes are labeled by the sizes m of the factors $M_m(\mathbb{C})$. In addition, the Bratteli diagram contains lines from each floor to the consecutive one, which fix the isomorphism class of the embedding $\Phi^{(n)}$ of $\mathcal{A}_i^{(n)}$ into $\mathcal{A}_i^{(n+1)}$; in our case, the lines are just the edges between nodes of $\mathcal{G}_q$, only now "source" and "range" of an edge are viewed as belonging to different floors. An example is shown in Figure 2.

Since the maps (2.24) are unital, the labeling of floors n with $n \geq 1$ follows from the dimensions of the factors of $\mathcal{A}_i^{(0)}$ and the structure of the (unlabeled) Bratteli diagram. The latter is the same for all $\mathcal{A}_i$, $i = 0, \ldots, K$, for $q = 2K+3$ fixed, except for the floors zero and one: It depends on the starting node i which other nodes can be reached on the first floor; among those, however, one always finds the node K, which in turn is connected to any other node of $\mathcal{G}_q$, thus the second floor already contains $K+1$ nodes.

It is clear from the description that the infinite Bratteli diagram $\mathcal{B}_{q,i}$ associated to the algebra $\mathcal{A}_i$ simply displays all the infinite paths on $\mathcal{G}_q$ with starting node i – which is why AF-algebras constructed in this particular way are also called path algebras [49].

In later sections, Bratteli diagrams (finite or infinite ones) will provide effective tools when we are interested in certain statements on algebras or homomorphisms thereof only up to isomorphism. One such statement follows immediately from the observation that, for fixed q, all the $\mathcal{B}_{q,i}$ look alike except for the first few floors: This implies that the algebras

$\mathcal{A}_i$ are mutually *stably isomorphic*, i.e. their infinite matrix amplifications $M_\infty(\mathcal{A}_i)$ are isomorphic – see e.g. [11].
Another fact which is obvious from the shape of the Bratteli diagrams is that all the $\mathcal{A}_i$ have trivial center. Therefore, the center of the global path observable algebra $\mathcal{A}$ is $\mathbb{C}^{K+1}$ – in accordance with the general theorems on the universal observable algebra $\mathcal{A}_{\mathrm{univ}}$ proven in [24].

2.3 Quasi-particle interpretation of paths

We have seen that our reformulation of the $c(2, q)$ highest weight modules as path spaces gives a neat picture of these CFTs, with a great deal of information encoded in one labeled graph, which moreover is the fusion graph of a distinguished sector of the theory. While in the remainder of this paper, we are mainly interested in the specific consequence that the path representations provide us with an operator algebraic description of global features of the $c(2, q)$ models, we here want to comment on other aspects of the path structure. We will give a natural interpretation of paths in terms of *quasi-particles* and use the decomposition of the modules $\mathcal{H}_i$ into sub-sectors of fixed quasi-particle numbers to compute the *characters*

$$\chi_i^K(q) = \mathrm{tr}_{\mathcal{H}_i}\, q^{L_0 - \frac{c}{24}} \tag{2.25}$$

of the $c(2, q)$ model with $q = 2K + 3$. (We hope there will be no confusion between the formal variable q in (2.25) and the q that labels the our conformal models.) It turns out that the quasi-particle picture of path spaces, details of which were first discussed in [56], leads to particular expressions for $\chi_i^K(q)$ that reveal interesting facts about the physics of the $c(2, q)$ models.

For simplicity, let us first concentrate on the example of the $c(2, 5)$ model, also known as the Lee-Yang edge singularity; this CFT describes a special critical point in the phase diagram of the Ising model with (complex) magnetic field [16]. The theory contains two irreducible modules with highest weights $h_0 = 0$ and $h_1 = -\frac{1}{5}$, which can be realized on the two-node tadpole graph $\mathcal{G}_5$, see Figure 1, with connectivity matrix

$$C_5 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} .$$

The label sequences $(l_m)_{m\geq 0}$ associated to the paths over $\mathcal{G}_5$ are sequences of 0's and 1's satisfying

$$l_m = 1 \quad \Longrightarrow \quad l_{m+1} = 0 \ , \tag{2.26}$$

the "tail condition" $l_m = 0$ for "large" m, and the initial conditions $l_0 = 1$ or $l_0 = 0$ for the vacuum or for the $h = -\frac{1}{5}$ representation, resp.
Here, we could directly interpret each sequence as a list of possible states of a quasi-particle, with a "1" in mth position indicating that the mth state is occupied; then (2.26) would be a generalized Pauli principle ("more exclusive" than the usual one) imposed on the quasi-particles. However, in the higher $c(2, q)$ models, it turns out that a slight change

of perspective is to be preferred. Namely, we identify a quasi-particle of the $c(2,5)$ with the basic length two segment $(1\,0)$ of a sequence. Then (2.26) becomes the ordinary Pauli principle, simply prohibiting that two quasi-particles "overlap". Except for that restriction (and the initial conditions on the sequences) the quasi-particles behave like free particles with a "dispersion relation" given by the energy operator $L_0^{\mathcal{G}}$ eq. (2.21).

Let us use this to compute the energies of all n-quasi-particle states, in the $h = -\frac{1}{5}$ representation, say. The state with the minimal energy corresponds to the sequence

$$s_0^{(n)} = (0; 1, 0, 1, 0, \ldots, 1, 0, 0, \ldots)$$

where the last "1" is at the $(2n-1)$ st position; because of (2.26), the quasi-particles cannot be packed more densely. $s_0^{(n)}$ has energy $E_0(n) = h_1 + 1 + 3 + \ldots + 2n - 1 = h_1 + n^2$. The simplest excitations of this n-quasi-particle ground state are obtained by shifting the last segment "10" to the right, step by step, and filling in "0"s. The resulting states have energies $E_0(n)+1$, $E_0(n)+2$, etc. If we excite the last two quasi-particles simultaneously (by shifting the last segment "1010" to the right), we obtain states with energies $E_0(n) + 2$, $E_0(n) + 4$, etc. The same can be done with $3,\ 4, \ldots, n$ quasi-particles. If we combine several of such shifts, we can generate all n-quasi-particle states from the ground state $s_0^{(n)}$, and thus, the total contribution of the n-quasi-particle sector $\mathcal{H}_1^{(n)}$ to the character is

$$\begin{aligned}
&\mathrm{tr}_{\mathcal{H}_1^{(n)}}\, q^{L_0-\frac{c}{24}} \\
&\qquad = q^{h_1-\frac{c}{24}} \cdot q^{n^2} \cdot \left(1 + q + q^2 + \ldots\right) \cdot \left(1 + q^2 + q^4 + \ldots\right) \cdots \left(1 + q^n + q^{2n} + \ldots\right) ;
\end{aligned}$$

the product occurs since the different excitations are independent of each other.

The total state space $\mathcal{H}_1$ can be decomposed with respect to the quasi-particle number,

$$\mathcal{H}_1 = \bigoplus_{n\geq 0} \mathcal{H}_1^{(n)} ,$$

which immediately leads to the following *sum form* for the character of the $h_1 = -\frac{1}{5}$ representation in the $c(2,5)$ minimal model:

$$\chi_1^1(q) = q^{h_1-\frac{c}{24}} \sum_{n\geq 0} \frac{q^{n^2}}{(q)_n}$$

where we have used the abbreviation $(q)_n := (1-q)(1-q^2)\cdots(1-q^n)$. For the vacuum module, corresponding to label sequences with $l_0 = 1$, the same procedure leads to the formula

$$\chi_0^1(q) = q^{h_0-\frac{c}{24}} \sum_{n\geq 0} \frac{q^{n^2+n}}{(q)_n} .$$

For the other $c(2,q)$ models, one may proceed in a similar fashion. Again, the Hilbert spaces (or the path spaces) can be decomposed with respect to the quasi-particle content

of the states, but now K different (and independent) species of quasi-particles are present; thus,

$$\mathcal{H}_i = \bigoplus_{n_1,\ldots,n_K} \mathcal{H}_i^{(n_1,\ldots,n_K)} \tag{2.27}$$

in the $c(2,q)$ model with $q = 2K+3$. The correspondence between quasi-particles and basic segments of paths (i.e. label sequences) is as follows: Since as in the $c(2,5)$ case, we want to interpret the difference two condition as an (ordinary) exclusion principle for quasi-particle, the latter must correspond to patterns of length two. Only now, each quasi-particle can occur in different "shapes". The "lightest" particle is given by the segment $(1\,0)$ as before, the second lightest occurs in the two forms $(2\,0)$ or $(1\,1)$, and so forth up to the "heaviest" quasi-particle which is the class of segments $(K\,0)$, $(K-1\,1)$, $\ldots$, $(1\,K-1)$. Expressions like "lightest" for the moment just refer to the dispersion law (2.21). Besides respecting the exclusion principle dictated by the difference two condition, this identification of quasi-particles with *classes* of patterns of length two ensures that each quasi-particle can be excited in energy steps of one.

It is amusing to look at an example (with $K \geq 2$) where a quasi-particle of the second lightest species upon excitation "moves through" one of the lightest type:

$$(0;2,0,1,0,\ldots) \hookrightarrow (0;1,1,1,0,\ldots) \hookrightarrow (0;1,0,2,0,\ldots) \hookrightarrow (0;1,0,1,1,\ldots) \hookrightarrow \ldots$$

Each arrow indicates that the total energy of the configuration increases by one unit, without changing the quasi-particle content; in the first step, the heavier quasi-particle changes its appearance from $(2\,0)$ to $(1\,1)$; after the second step, it has passed the lighter quasi-particle, which then has "jumped" to the left – a phenomenon reminding of the "time shift" in soliton scattering.

This sketch already suggests how to implement the Virasoro modes L_n on the path spaces in a way which takes the path structure into account. In [56], formulas for the su(1,1) generators $L_{\pm 1}$ acting in the vacuum module have been constructed, following the guiding principle that these operators should leave the decomposition (2.27) of $\mathcal{H}_i$ invariant. (In contrast, the L_n one would obtain directly from the PBW vectors do not respect the quasi-particle numbers.) To state the precise formulas for $L_{\pm 1}$, and also to rigorously prove that the quasi-particles of different species can be excited independently of each other, is slightly technical, and we refer to [56] for the complete analysis.

Once these results are established, it is again straightforward to derive sum forms for the characters of the $c(2,q)$ model, $q = 2K+3$:

$$\chi_i^K(q) = q^{h_i - \frac{c}{24}} \sum_{n_1,\ldots,n_K \geq 0} \frac{q^{N_1^2+\cdots+N_K^2+N_{i+1}+\cdots+N_K}}{(q)_{n_1}\cdots(q)_{n_K}} \tag{2.28}$$

where $N_i := n_i+\ldots+n_K$. The denominators show up because the different quasi-particles can be excited independently, the q-exponent in the numerator is the minimal energy of a configuration with n_l quasi-particles of species l, $l = 1,\ldots,K$.

One can compare these formulas to the well-known Rocha-Caridi expressions for the characters of Virasoro minimal models, which follow directly from the Feigin-Fuchs results

on the chain of singular vectors. If one applies the Jacobi triple product identity to the Rocha-Caridi characters, see [41], one obtains the *product form*

$$\chi_i^K(q) = \; q^{h_i - \frac{c}{24}} \prod_{\substack{l \not\equiv 0, \pm(i+1) \, (\mathrm{mod}\, 2K+3) \\ l>0}} (1-q^l)^{-1} \; . \tag{2.29}$$

Equating (2.28) and (2.29) yields combinatorial identities known as *Andrews-Gordon identities*, see e.g. [5]. In the special case of $K = 1$, they reduce to the famous Rogers-Ramanujan identities.

The surprising feature of the character sum formulas (2.28) is their relation to non-conformal models. In the first place, expressions of the same type naturally appear in the theory of *1-dimensional quantum spin chains*. This was first shown in [39], where partition sums of such chains were calculated from the Bethe Ansatz. Kedem and McCoy also were the first to realize that the excitation spectrum of the chains can be interpreted in terms of quasi-particles, having specific dispersion relations and obeying generalized Pauli principles.
Since 1-dimensional quantum spin chains are essentially equivalent to 2-dimensional statistical models on the lattice, the results of [56] on path space representations of $L_{\pm 1}$ are also relevant as an approach towards a Virasoro action on the lattice.

Conformal field theories arise in the continuum limit of 2-dimensional lattice models (or 1-dimensional quantum spin chains) at the critical point. Alternatively, they can appear as scaling limits of *2-dimensional massive QFTs* – in particular of integrable field theories which have been obtained as perturbations of CFTs [66]. It turns out that the conformal characters (2.28) carry information on such perturbations as well:
The q-exponent, giving the ground state energy of a quasi-particle configuration with prescribed $\mathtt{n} := (n_1, \ldots, n_K)$, can be written as a quadratic form $\mathtt{n}^{\mathrm{t}} M_q \mathtt{n} + m_{q,i}^{\mathrm{t}} \mathtt{n}$ with an integer $K \times K$ matrix M_q and integer K-vectors $m_{q,i}$. Whereas M_q is "universal" within the $c(2,q)$ model, $m_{q,i}$ also depends on the sector.
In our cases, M_q is the inverse of the Cartan matrix of the K-node tadpole graph. The associated connectivity matrix has a Perron-Frobenius eigenvector v_q, and the astonishing fact is that the ratios of the entries of v_q coincide with the mass ratios of the particles present in the so-called $\phi_{1,3}$-perturbation of the $c(2,q)$ minimal model, see [25]. In this sense, the expressions "lightest" and "heaviest" quasi-particle used above receive a literal meaning.

Note that such "coincidences" occur for other sum expressions of conformal characters, too: By now, sum formulas for the characters of many conformal coset models have been found (without reference to path representations), see e.g. [61, 38, 10], and also [41] for a discussion W-algebra extensions of certain minimal models. Most remarkably, there exist two different sum forms for the characters of the Ising models, involving the inverse of the Cartan matrix of A_1 or of E_8, respectively. On the other hand, the two possible massive perturbations of the Ising CFT have either one or eight massive particles, in the latter case with mass ratios given by the Perron-Frobenius vector of the incidence matrix of the E_8 graph [66].

In summary, we have observed that the quasi-particle structure of the CFT highest weight representations and the associated sum forms of the conformal characters reveal certain aspects of non-conformal deformations (lattice models or massive QFTs) of the CFT. Although the precise relationship remains to be worked out, we may at least conclude that the path representations of the $c(2,q)$ models are not just an artifact of our constructions, but do indeed have deeper physical meaning.

3. The quantum symmetry algebra

For our construction of global amplimorphisms for the $c(2,q)$ models, we will make use of an action of a quantum symmetry algebra (QSA) on the path representations, which allows to determine covariant field multiplets. In section 3.1, we collect some general statements on "quantum symmetries" of low-dimensional QFTs. We do, however, in no way attempt to give a complete account of this subject, which has been an area of intense research during the last decade. We recommend e.g. [29, 58] as sources where to find details and references to the historical development. Likewise, nothing new will be added to the general theory of quantum symmetries – except for the remark that for so-called quasi-rational CFTs there exists a canonical semi-simple QSA, a fact that is somewhat contrary to the prevailing opinion. Section 3.2 is devoted to these matters, but it can be skipped if one is merely interested in the QSA action on the path spaces, which is set up in section 3.3.

3.1 General remarks on quantum symmetries

Local quantum field theories on a low-dimensional space-time are interesting especially because of their superselection structure. Their statistics is governed by the braid group in contrast to the permutation group symmetry of QFTs in higher dimensions. The statistical data of a QFT define a representation category, which is a "rigid braided (or symmetric) monoidal category with unit", see e.g. [46, 29]. A natural question to ask is whether this category is equivalent to the representation category of some group or algebra, which then could be regarded as the internal symmetry group (or algebra), the "global gauge group", of the model. For local QFTs in space-time dimension ≥ 3, this problem has been settled by the famous duality theorem of Doplicher and Roberts [19] with the result that in this situation the statistical properties always "come from" a compact Lie group.

In two dimensions, and for charges localized in space-like cones in $2+1$ dimensions, the situation is not quite as clear yet. Research essentially focuses on three different variants of quantum symmetry algebras:

One is given by *quantum groups* in the sense of deformations of the universal enveloping algebras of ordinary Lie algebras. In certain theories, e.g. the WZW models, they arise rather naturally [4, 1], and they have the further virtue that they are closely related to lattice discretizations of CFTs, see e.g. [51]. It is, however, rather unlikely that quantum groups cover all kinds of statistics that can arise in conformal models. In addition, there is the unpleasant feature that in the cases most relevant for rational CFTs, namely when

the deformation parameter is a root of unity, indecomposable representations show up, the significance of which remains to be uncovered.

A more speculative, and more spectacular, concept of quantum symmetry uses *Ocneanu's string algebras* or *"paragroups"* [49]. They occur naturally in the general DHR framework as intertwiner algebras associated to the endomorphisms of a local QFT [24, 28, 54]. It is clear that these string algebras contain all information on the statistical properties of a theory, but a "naive" implementation of string algebras as "global gauge symmetries" of a QFT leads to huge total Hilbert spaces and enormous field algebras, as was shown in [53]. The ideas in [24, 59] suggest that an appropriate use of string algebras as symmetry algebras requires to give up the clear-cut division between space-time and internal symmetries. Such a "mixing" of symmetries seems to be realized in our $c(2, q)$ models, since as C^*-algebras, the intertwiner algebras (internal symmetries) are simply path algebras associated to the fusion graphs of the QFT, and thus they are of the same kind as our path observable algebra (space-time symmetries). It should be interesting to pursue this relation further, but at present, we are interested in quantum symmetries as a mere practical tool and will therefore not resort to string symmetry algebras either.

Finally, a comparatively modest approach to quantum symmetry is to rely on *semi-simple* **-algebras* with additional structures making them into *weak quasi-triangular quasi-Hopf algebras* [45]. It has been shown in [58] that under certain standard assumptions such a semi-simple QSA always exists, in particular for rational CFTs. One could argue that this type of QSA does not always arise in a natural way, and that it may be difficult to compute the extra structure they are supplied with. Even more problematical, the constructions known so far did not single out one specific semi-simple QSA for a given model, not even up to isomorphism. Instead, if $i \in I$ labels the sectors of a rational CFT, say, with fusion rules N_{ij}^k, then each algebra

$$\mathsf{g} \cong \bigoplus_{i\in I} M_{n_i}(\mathbb{C}) \tag{3.1}$$

can be viewed as a quantum symmetry of the CFT as soon as the integers n_i satisfy the inequalities [58]

$$n_i\, n_j \geq \sum_{k\in I} N_{ij}^k\, n_k \ . \tag{3.2}$$

We will, however, show in subsection 3.2 that for a wide class of CFTs the results of [48] yield a canonical set of such integers n_i. Since, moreover, in our special path setting the (canonical) semi-simple QSA leads to a very natural (and "slim") field algebra, and since we will not need the more complicated additional structures on g for our purposes, it is this third concept of quantum symmetry that will be used in the following.

Let us just sketch the main data of semi-simple QSAs g and how they are implemented into the space of states of a CFT. Complete definitions and proofs can be found in [58].
g is a matrix algebra as in (3.1) with $n_0 = 1$ for the vacuum sector. Consequently, the projection onto the factor $M_{n_0}(\mathbb{C})$, $\varepsilon \equiv \mathrm{pr}_0 : \mathsf{g} \longrightarrow \mathbb{C}$, is the natural candidate for the *co-unit* of g; it has to be checked whether this ε satisfies the correct relations.
Since the representation category of g must "mimic" the braided tensor category associated to the CFT, the QSA g has to be endowed with a *co-product* $\Delta : \mathsf{g} \longrightarrow \mathsf{g} \otimes \mathsf{g}$ which

"reproduces" the fusion rules of the CFT. The simplest way to make this requirement precise is to look at the (two-floor) Bratteli diagram of Δ viewed as an algebra homomorphism: The diagram must contain N_{ij}^k lines from the factor $\mathsf{g}_k \equiv e_k \cdot \mathsf{g}$ of g to the factor $\mathsf{g}_i \otimes \mathsf{g}_j$ of $\mathsf{g} \otimes \mathsf{g}$. Here, e_i are the minimal central projections of g, i.e. $e_i \cdot \mathsf{g} \cong M_{n_i}(\mathbb{C})$. The formulation with the help of Bratteli diagrams has the additional advantage to make the freedom of so-called *"twists"* [20] explicit: Δ is fixed only up to conjugation with a unitary in $\mathsf{g} \otimes \mathsf{g}$. Of course, the other data of g have to be changed accordingly when Δ is twisted, since there are compatibility conditions; e.g. co-unit and co-product must obey $(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta$.

Since $N_{0j}^k = \delta_{kj}$ for all k, the co-product is always injective, but it need *not* be a *unital* embedding. Obviously, Δ is unital iff (3.2) holds as an equality for all $i, j \in I$.

Similarly, the co-product in general is not co-associative (it cannot be if it is non-unital) but only quasi-co-associative, i.e. there exists a (quasi-invertible) *re-associator* $\varphi \in \mathsf{g} \otimes \mathsf{g} \otimes \mathsf{g}$ intertwining $(\Delta \otimes \mathrm{id}) \circ \Delta$ and $(\mathrm{id} \otimes \Delta) \circ \Delta$.

The braid group statistics of a low-dimensional QFT is reflected by the existence of an *R-matrix* $R \in \mathsf{g} \otimes \mathsf{g}$ which is an intertwiner between Δ and Δ', the co-product with tensor factors interchanged.

We have ignored the *antipode* $S : \mathsf{g} \longrightarrow \mathsf{g}$, a $\mathbb{C}$-linear anti-automorphism of g which translates charge conjugation of sectors into the QSA, and have also not discussed important compatibility conditions like the so-called *pentagon* and *hexagon identities*, involving Δ, φ and R. Sometimes, existence of a representation of the modular group on g is also assumed, see e.g. [32, 62].

Since we will not use those structures, we again refer to [58] for further details and merely recall that given a rational CFT with fusion rules N_{ij}^k and a set of integers n_i, $i \in I$, $n_0 = 1$, satisfying (3.2), then one can solve all the constraints and obtain a weak quasi-triangular quasi-Hopf algebra as a QSA which reproduces the representation category of the CFT.

The QSA is implemented as follows: One forms an enlarged Hilbert space $\mathcal{H}^{\mathrm{tot}}$ in which each irreducible representation space $\mathcal{H}_i$, $i \in I$, of the observable algebra occurs with multiplicity n_i, i.e.

$$\mathcal{H}^{\mathrm{tot}} = \bigoplus_{i \in I} \mathcal{H}_i \otimes V_i \ , \qquad \dim V_i = n_i \ . \tag{3.3}$$

Each multiplicity space V_i carries an irreducible representation of g which is equivalent to the defining representation $\tau_i \equiv \mathrm{pr}_i$ on $\mathbb{C}^{n_i}$ – assuming that g is directly given as matrix algebra. Thus, $\mathcal{H}^{\mathrm{tot}}$ carries a representation of g, which will be denoted by U.

What we have sketched is already a special realization of the QSA within a QFT, of the form constructed in [58]. There it was shown that $\mathcal{H}^{\mathrm{tot}}$ furthermore carries a *field algebra* $\mathcal{F}$ (with a net structure inherited from $\mathcal{A}$), which can be decomposed into *fields multiplets* transforming covariantly under the g-action. The observable algebra $\mathcal{A}$ is recovered as the fixed point algebra of $\mathcal{F}$. The Hamiltonian of the QFT also commutes with the g-action. Moreover, the local braid relations of the field operators can be written in terms of the R-matrix of g, and in this way one gains complete control of the braid group representations associated with the sectors of a low-dimensional QFT.

The covariance properties of field multiplets will be an important part of our construction of global amplimorphisms of the observable algebra. They are formulated in eq. (4.10)

below, in a slightly different form than in [58]. Beyond that, we will only use information encoded in the co-product of the QSA, and of course the representation U of g on $\mathcal{H}^{\mathrm{tot}}$.

First of all, we have to choose an appropriate matrix algebra $\mathsf{g}_{(q)}$ as a QSA for each of our $c(2,q)$ minimal models, i.e. we have to fix the sizes n_i in eq. (3.1). We claim that the choice

$$n_i = i+1 \tag{3.4}$$

is one possibility; in order to show this, we prove the following:

Lemma 3.1 The integers $n_i = i+1$, $i = 0, \ldots, K$ satisfy the inequalities

$$n_i\, n_j \geq \sum_k N_{ij}^k\, n_k \ ,$$

where N_{ij}^k are the fusion rules of the $c(2,q)$ model, $q = 2K+3$, as listed in eqs. (2.4,5).

Proof: Suppose that $i \geq j$. Then eqs. (2.4,5) give

$$\begin{aligned}\sum_k N_{ij}^k\, n_k &= n_{i-j} + n_{\{i-j+2\}} + \ldots + n_{\{i+j\}} \\ &= i-j+1 \ + \ \{i-j+2\}+1 \ + \ \ldots \ + \ \{i+j\}+1 \\ &\leq (i+1)(j+1) = n_i\, n_j \ ,\end{aligned}$$

where in the third line we have used the estimate $\{m\} \leq m$ before summing up. ∎

Therefore, the results collected above imply that

$$\mathsf{g}_{(q)} := \mathbb{C} \oplus M_2(\mathbb{C}) \oplus \ldots \oplus M_{K+1}(\mathbb{C}) \tag{3.5}$$

can be endowed with all the structures making it into *some* QSA of the $c(2,q)$ model, $q = 2K+3$, and we could immediately proceed with implementing $\mathsf{g}_{(q)}$ on the total Hilbert space

$$\mathcal{H}^{\mathrm{tot}}_{(q)} := \mathcal{H}_0 \oplus \left(\mathcal{H}_1 \otimes \mathbb{C}^2\right) \oplus \ldots \oplus \left(\mathcal{H}_K \otimes \mathbb{C}^{K+1}\right) \ . \tag{3.6}$$

But before that, we would like to take the opportunity and argue that the matrix algebra (3.5) can indeed be called *the* semi-simple QSA of the $c(2,q)$ model.

3.2 The canonical choice of sector multiplicities

In order to explain that the choice of dimensions $n_i = i+1$ in eq. (3.4) is even a *canonical* one, we will slightly digress and give an account of some results of the work [48]. There, the notion of *quasi-rational CFTs* was introduced: These are models of CFT such that a certain factor space of each irreducible highest weight module is finite-dimensional. The dimensions n_i of those factor spaces are invariants of the theory, they satisfy the sub-multiplicativity relation (3.2) and can, therefore, be used as dimensions of the defining representation of a semi-simple quantum symmetry algebra of the model. We will show

that the minimal models of our interest are quasi-rational in Nahm's sense, and that the dimensions of the relevant factor spaces are given by formula (3.4).

Up to the present, the only treatment of quasi-rationality available is in terms of modes of unbounded quantum fields, and using the fusion product as a computational tool. It would be very interesting to try and fit the notions developed in [48] into the algebraic DHR framework, and to find out whether there is a more conceptual interpretation of the dimensions n_i, perhaps by some operator algebraic constructions. In view of the inequalities (3.2), it seems likely that the n_i are related to the Jones indices d_i^2 associated to the sectors, since the statistical dimensions d_i satisfy (3.2) as equalities, see e.g. [24, 43]. On the other hand, it is quite clear already from [48] that quasi-rationality is a very useful property for practical problems: It allows, in particular, for an algorithmic definition of fusion in a large class of CFTs. Below, we will add further remarks on this aspect.

The notion of a quasi-rational representation can be introduced for arbitrary bosonic W-algebras with a finite number of generating fields. A rather detailed definition of W-algebras can be found e.g. in [46, 12], but for our purposes, the following sketch is sufficient: $\mathcal{W}$ contains a finite set of generating fields $W^s(z) = \sum_{n\in\mathbb{Z}} W^s_n\, z^{-n-s}$ with conformal dimensions $s \in \mathbb{Z}_+$, where $z \in \mathbb{C}$ is the coordinate of left-moving fields, and the rhs gives the expansion of $W^s(z)$ in terms of Laurent modes of $W^s(z)$. Among them, there is a field $W^2(z)$ of dimension two, which we identify with the energy momentum tensor $T(z)$; all the other generators $W^s(z)$ are Virasoro primary. $\mathcal{W}$ is then linearly generated by these fields, their derivatives with respect to z, and (derivatives of) normal-ordered products. Note that one can choose a linear basis of $\mathcal{W}$ consisting only of primary generators $W^s(z)$ and quasi-primary normal ordered products. With respect to this basis, $\mathcal{W}$ is an infinite-dimensional Lie algebra, and for most of what follows, we may simply regard a W-algebra as the universal enveloping algebra generated by the Laurent modes of the fields $W^s(z)$ – in this way avoiding a general discussion of normal ordered products.

We denote by $\mathcal{W}_{--}$ the linear span of modes

$$\begin{aligned} \mathcal{W}_{--} &:= \left\{ \phi_n \,|\, \phi(z) \in \mathcal{W},\ \phi \neq \mathbf{1},\ n \leq -\dim\phi \right\} \\ &= \left\{ \oint \omega(z)\phi(z) \,|\, \phi(z) \in \mathcal{W},\ \phi \neq \mathbf{1},\ \omega \text{ a 1-form vanishing at } \infty \right\}, \end{aligned} \tag{3.7}$$

where in the second description we have used contour integration around zero to project onto Laurent modes.

Definition 3.2 Let V be an irreducible highest weight representation of a finitely generated bosonic W-algebra $\mathcal{W}$, and put $V^{\mathrm{a}} := \mathcal{W}_{--}V$. The representation V is called *quasi-rational* if the quotient space V/V^{a} is finite-dimensional. A CFT with finitely generated bosonic W-algebra is called *quasi-rational* if all the irreducible highest weight representations making up the chiral space of states are quasi-rational.

Note that in the last part of the definition we do *not* require that the CFT involves only a finite number of irreducible representations, i.e. that it is rational. On the one hand, there is the plausible conjecture that any rational CFT with appropriate W-algebra is quasi-rational. On the other hand, there are definitely examples of non-rational quasi-rational

theories – which is one of the advantages of working with Definition 3.2. In particular, many of the $N = 2$ superconformal QFT associated to Calabi-Yau manifolds probably are non-rational, but since they are relevant for string theory, it is important to have tools which work in a larger class of CFT than just the rational ones. The restriction to bosonic W-algebras in Definition 3.2 is not essential, see [34] for an extension to the fermionic case.

Given a representation V_i of a quasi-rational CFT, we use the notation

$$n_i = \dim \left(V_i / V_i^{\mathrm{a}} \right) \tag{3.8}$$

for the dimension of the factor space. Note that $n_i \geq 1$ for any representation. To a subspace $V_i^{\mathrm{s}} \subset V_i$ such that $\dim V_i^{\mathrm{s}} = n_i$ and $V_i^{\mathrm{s}} + V_i^{\mathrm{a}}$ is dense in V_i, we will refer to as a *small space* of the CFT. While the integers n_i are invariantly defined, the small spaces are not. Often, however, there are natural choices in explicit computations.

The most interesting property of the dimensions n_i was also proven in [48]:

Proposition 3.3 [48] Consider a CFT with W-algebra $\mathcal{W}$ and let V_m, $m \in I$, be the collection of irreducible $\mathcal{W}$-representations occurring in the CFT. Assume that among these there are two quasi-rational highest weight representations V_i and V_j with small space dimensions n_i and n_j. Denoting the fusion rules of the theory by N_{ij}^k for $i, j, k \in \mathcal{I}$, we have the following inequality:

$$n_i \, n_j \geq \sum_k N_{ij}^k \, n_k$$

Corollary 3.4 The set of quasi-rational highest weight representations of a W-algebra forms a sub-category (of the category of all highest weight representations) which is closed under fusion.
Quasi-rational representations are semi-rational, i.e. the fusion decomposition of two quasi-rational representations contains only a finite number of irreducibles.
The n_i can be used as dimensions of the defining representation of a semi-simple quantum symmetry algebra of a quasi-rational CFT.

To sketch the proof of Proposition 3.3, we have to use the *fusion product* of two W-algebra representations. For $\alpha = 1, 2$, choose two distinct "punctures" $z_\alpha \in \mathbb{C}$ and "insert" a representation V_{i_α} at z_α. The action of a field $\phi(z) \in \mathcal{W}$ on a vector $v_\alpha \in V_{i_\alpha}(z_\alpha)$ can be written as a contour integral

$$\oint_{C_\alpha} \omega(z) \phi(z) \, v_\alpha$$

where $\omega(z)$ is a meromorphic 1-form and C_α is a "small" contour encircling z_α. The well-known fusion product representation of $\mathcal{W}$ on the tensor product $V_{i_1} \otimes V_{i_2}$ is defined via Cauchy's formula

$$\oint_{C_{12}} \omega(z) \phi(z) \, (v_1 \otimes v_2) := \Big(\oint_{C_1} \omega(z) \phi(z) \, v_1 \Big) \otimes v_2 + v_1 \otimes \oint_{C_2} \omega(z) \phi(z) \, v_2 \; , \tag{3.9}$$

where C_{12} encircles both z_1 and z_2. Introducing a mode expansion on the lhs – here, a choice is involved – leads to a z_α-dependent, co-product-like expression of the fusion product action of ϕ_n on $V_{i_1} \otimes V_{i_2}$ in terms of modes ϕ_m that act in each V_{i_α} separately, see e.g. [23, 46, 48]. If the action (3.9) can be diagonalized we obtain a decomposition into irreducibles – hence the name "fusion product".

Contour integrals may be used to prove Proposition 3.3 in the following way. Consider any vector $v_1 \otimes v_2 \in V_{i_1} \otimes V_{i_2}$ such that $v_\alpha \in V_{i_\alpha}$ are L_0-eigenvectors. Having chosen small spaces $V^{\mathrm{s}}_{i_\alpha}$, we can decompose v_2 into a sum $v_2 = v_2^{\mathrm{s}} + v_2^{\mathrm{a}}$ with $v_2^{\mathrm{s}} \in V^{\mathrm{s}}_{i_2}$, $v_2^{\mathrm{a}} \in V^{\mathrm{a}}_{i_2}$. By definition, v_2^{a} is of the form $v_2^{\mathrm{a}} = \phi_{-h-n} v_2'$ for some $v_2' \in V_{i_2}$ and some field mode $\phi_{-h-n} \in \mathcal{W}_{--}$, where h is the dimension of ϕ and $n \geq 0$. Using eq. (3.9), this can be rewritten as

$$v_1 \otimes v_2 = \phi_{-h-n}(v_1 \otimes v_2') - \Big(\sum_{l \geq 0} f_l(z_1 - z_2) \phi_{l-h+1}\, v_1 \Big) \otimes v_2' \tag{3.10}$$

where $f_l(z_1 - z_2)$ are certain meromorphic functions arising from the mode expansion. The first term on the rhs of (3.10) is in $\big(V_{i_1} \otimes V_{i_2}\big)^{\mathrm{a}}$, and the remaining terms have a lower L_0-degree than $v_1 \otimes v_2$. Proceeding inductively, and treating v_1 in the same way, one finally reaches vectors in the tensor product of the small spaces, which proves that $V_{i_1} \otimes V_{i_2}$ is generated by vectors in $\big(V_{i_1} \otimes V_{i_2}\big)^{\mathrm{a}}$ and $V^{\mathrm{s}}_{i_1} \otimes V^{\mathrm{s}}_{i_2}$. In other words, the space $V^{\mathrm{s}}_{i_1} \otimes V^{\mathrm{s}}_{i_2}$ contains a small space $\big(V_{i_1} \otimes V_{i_2}\big)^{\mathrm{s}}$ of $V_{i_1} \otimes V_{i_2}$ endowed with the $\mathcal{W}$-action from the fusion product. By definition, this representation contains all other irreducible highest weight representations V_k with multiplicity $N^k_{i_1 i_2}$ which implies the desired inequality. For more details of the proof, we refer to [48]. We would, however, like to emphasize that the statement in Proposition 3.3 is independent on any choice to be made when giving explicit formulas for the fusion product action of field modes.

The practical use of the concept of quasi-rationality combined with the fusion product now becomes apparent: It allows to determine the fusion rules of quasi-rational representations by a finite algorithm. One simply has to diagonalize the zero-mode sub-algebra $\mathcal{W}_0$ of $\mathcal{W}$ on the finite-dimensional space $V^{\mathrm{s}}_{i_1} \otimes V^{\mathrm{s}}_{i_2}$ in order to obtain a decomposition into irreducibles. In this procedure, however, two subtleties are hidden. First of all, diagonalizability may fail to hold in general, although the known examples for such a situation are typically plagued with certain pathological features. The other problem involves the so-called *spurious states*, i.e. vectors in the space $V^{\sigma}_{i_1 i_2} := \big(V^{\mathrm{s}}_{i_1} \otimes V^{\mathrm{s}}_{i_2}\big) \cap \big(V_{i_1} \otimes V_{i_2}\big)^{\mathrm{a}}$. This intersection is non-zero if the inequality in Proposition 3.3 becomes strict, e.g. in most minimal models of the Virasoro algebra. Fusion rules are then obtained from diagonalizing the zero mode action in the space $\big(V^{\mathrm{s}}_{i_1} \otimes V^{\mathrm{s}}_2\big)/V^{\sigma}_{i_1 i_2}$. The construction of spurious states is not yet well understood at an abstract level, although in concrete examples it is usually possible to determine $V^{\sigma}_{i_1 i_2}$ from the null-fields of the theory.

As was shown in [48], the fusion rules can be calculated – up to the subtleties mentioned above – on even smaller spaces: Denote by V^{h}_i the highest weight subspace (wrt L_0) of the module V_i, then one can show in a similar fashion as before that both $V^{\mathrm{h}}_{i_1} \otimes V^{\mathrm{s}}_{i_2}$ and $V^{\mathrm{s}}_{i_1} \otimes V^{\mathrm{h}}_{i_2}$ contain the space $\big(V_{i_1} \otimes V_{i_2}\big)^{\mathrm{h}}$. This further reduction is very useful in concrete applications, and often allows to avoid cumbersome calculations with spurious states.

We will now show that the minimal models with central charge $c(2,q)$ and highest weights h_i as in eqs. (2.2,3) are quasi-rational CFTs and that the dimensions of their small spaces are given by the simple formula (3.4).
Consider a highest weight representation with highest weight

$$h_{r,s} \equiv h_{r,s}(p,q) = \frac{(pr-qs)^2-(p-q)^2}{4pq} \tag{3.11}$$

of the Virasoro algebra at central charge $c(p,q)$, where $r,s \geq 1$ are integers, but we admit arbitrary $p,q \in \mathbb{R}_{>0}$. Feigin and Fuchs have shown that the Verma module $\mathcal{V}_h$ over $|h_{r,s}\rangle$ contains a singular vector $|v\rangle$ at level $r \cdot s$ (and maybe others, all at higher energy), which is of the form

$$|v\rangle = L_{-1}^{rs}|h_{r,s}\rangle + \alpha|v'\rangle \tag{3.12}$$

where the complex number α depends on r,s and p/q, and $|v'\rangle \in \mathcal{V}_h$ involves the modes L_{-2}, $L_{-3},\ldots$ – see [21] for more explicit statements. After passing to the irreducible module $\mathcal{H}_{r,s}$, eq. (3.12) implies that

$$L_{-1}^{rs}|h_{r,s}\rangle \in \big(\mathcal{H}_{r,s}\big)^{\mathrm{a}}$$

since L_{-2}, $L_{-3},\ldots \in \mathrm{Vir}_{--}$, cf. eq. (3.7). On the other hand, one can easily convince oneself – using, e.g., the explicit formula for normal ordered products in [47, 12] – that none of the vectors $|h_{r,s}\rangle$, $L_{-1}|h_{r,s}\rangle,\ldots,L_{-1}^{rs-1}|h_{r,s}\rangle$ can be written as $\phi_m|v''\rangle$ for some $|v''\rangle \in \mathcal{H}_{r,s}$, some normal ordered product $\phi(z)$ of the energy momentum tensor, and $m \leq -\dim\phi$. Thus, for the degenerate models of the Virasoro algebra with highest weights $h_{r,s}(p,q)$, the vectors listed above can be taken as a basis of the small spaces; the equation

$$n_{r,s} \equiv \dim\big(\mathcal{H}_{r,s}\big)^{\mathrm{s}} = rs$$

for arbitrary degenerate representations and in particular formula (3.4) for our models follow. As a further consequence, we indeed obtain the sub-multiplicativity relation in Lemma 3.1 as a special case of Proposition 3.3, without resort to any combinatorics.

The models with $h_{r,s}(p,q)$ as above and $r,s \in \mathbb{Z}_+$, $p/q \notin \mathbb{Q}$, are simple examples of non-rational quasi-rational CFTs, and of theories where the dimensions $n_{r,s}$ satisfy the relation of Proposition 3.3 as an equality. In contrast, for the minimal models $c(2,q)$, $q \geq 5$ an odd integer, we have $n_in_j > \sum_k N_{ij}^k n_k$ in general, which also means that there exist spurious states.

As mentioned above, this fact complicates the computation of fusion rules by diagonalizing the L_0-action on $\mathcal{H}_i^{\mathrm{s}} \otimes \mathcal{H}_j^{\mathrm{s}}$ or $\mathcal{H}_i^{\mathrm{s}} \otimes \mathcal{H}_j^{\mathrm{h}}$. Nevertheless, the special cases we are interested in are simple enough to let us circumvent these problems, e.g. in the following three ways:
First of all, we may "perturb" the models slightly by moving the central charge away from the minimal values $q \in \mathbb{Z}$ while keeping the relations (3.11) for the highest weights. Then, the model becomes merely degenerate, the spurious states disappear, and we may read off the N_{ij}^k from the L_0-action in the tensor product of small spaces. Afterwards, we can move c back to the minimal value, take into account the conformal grid symmetry $h_{r,s} = h_{q-r,p-s}$

for minimal models if necessary, and we will recover the fusion rules of the minimal model. In the limit $c \to c_{\mathrm{min.mod.}}$ some higher level vectors in the irreducible components of the fusion product move towards singular vectors of the minimal representations, thus producing the spurious states in a controlled way.

The second possibility is to calculate the fusion rules directly within the minimal model $c(2,q)$, but using diagonalization of L_0 in the smaller spaces $\mathcal{H}_i^{\mathrm{s}} \otimes \mathcal{H}_j^{\mathrm{h}}$. All the highest weight spaces are, of course, one-dimensional in pure Virasoro models. For each q, the small space $\mathcal{H}_1^{\mathrm{s}}$ of the highest weight representation with $h_1 = -\frac{q-3}{2q}$ is two-dimensional, so we can obtain the fusion rules of $\mathcal{H}_1$ with $\mathcal{H}_k,\ k \neq 0$, from diagonalizing 2×2 matrices. But in each $c(2,q)$ model, the conformal family corresponding to $\mathcal{H}_1$ generates the whole fusion ring.

The most direct method, however, is to calculate the spurious states explicitly. In the $c(2,q)$ models, this can be done since the null-fields are known, see eq. (2.9). Their Laurent modes applied to highest weight states give null-vectors which can be used to compute the spurious states via contour integration and Cauchy's theorem. We refrain from giving further details here and rather refer to [48, 3].

In summary, we have seen that Nahm's results [48] indeed provide a *canonical* choice for the isomorphism type of a *semi-simple quantum symmetry algebra* for a quasi-rational CFT, namely

$$\mathsf{g} = \bigoplus_{i \in \mathcal{I}} M_{n_i}(\mathbb{C})$$

where $n_i = \dim \mathcal{H}_i^{\mathrm{s}}$ are the small space dimensions associated to the irreducible representations. The n_i are *invariants* of the quasi-rational CFT.

One can show that for quasi-rational models even the other basic data of g, namely the co-product, the R-matrix and the re-associator, can be reconstructed explicitly: The small spaces can be used to define a finite-dimensional vector bundle equipped with a flat connection, which leads to a generalization of the Knizhnik-Zamolodchikov equation from WZW models to arbitrary quasi-rational CFTs. Then, Drinfeld's construction [20] can be applied to recover all data of a weak quasi-triangular quasi Hopf algebra from the generalized Knizhnik-Zamolodchikov equation [3].

The fusion rules are part of this structure, to be determined by the diagonalization procedure sketched above. Therefore, we can in principle delete them from the input data listed in section 2.1 and instead derive them from information referring only to individual highest weight representations.

3.3 Action of the quantum symmetry algebra on the path spaces

Irrespective of whether the sector multiplicities $n_i = i+1$ chosen in eq. (3.4) are canonical or not, we will see that they lead to a particularly natural action of the associated quantum symmetry algebra $\mathsf{g}_{(q)}$ on the "amplified" Hilbert space $\mathcal{H}_{(q)}^{\mathrm{tot}}$, see eqs. (3.5,6), of the minimal model with central charge $c(2,q)$, $q = 2K+3$. The point is that for this special choice of n_i, the space $\mathcal{H}_{(q)}^{\mathrm{tot}}$ can again be represented as a path space over a Bratteli diagram $\widehat{\mathcal{B}}_q$ that has essentially the same form as the $\mathcal{B}_{q,i}$ underlying the individual spaces $\mathcal{H}_i$.

The extended Bratteli diagram $\widehat{\mathcal{B}}_q$ looks as follows: Floors are numbered $-2,\ -1,\ 0,\ 1,\ \ldots$; the -2 nd floor consists of one node, labeled $*$, all other floors of $K+1$ nodes labeled $0,\ldots,K$ as usual; the embedding matrices between floors l and $l+1$ for $l \geq -1$ are the $(K+1)\times(K+1)$ connectivity matrices C_q of the $c(2,q)$-graph $\mathcal{G}_q$, see eq. (2.7), whereas the matrix describing the embedding of floor -2 into floor -1 is simply $C_{\rm in} = (1,1,\ldots,1)^{\rm t} \in M_{(K+1)\times 1}(\mathbb{Z})$ – i.e. the node $*$ is joined to every node on the -1 st floor by a single edge. Compared to the Bratteli diagrams $\mathcal{B}_{q,i}$ of section 2.2, the only new building block of $\widehat{\mathcal{B}}_q$ is an extremely simple, "canonical" one. In Figure 2, an example is shown.

Let $\widehat{\mathcal{P}}^{(n)}_{*,i}$ denote the space of all paths on $\widehat{\mathcal{B}}_q$ from the node $*$ on the -2 nd floor to the node i on the n th floor, for $n \geq -1$ and $i = 0,\ldots,K$.

Lemma 3.5 $\dim \widehat{\mathcal{P}}^{(0)}_{*,i} = n_i$

Proof: We compute the dimension of this path space by applying the first two embedding matrices to the dimension 1 of the space of length zero paths:

$$\dim \widehat{\mathcal{P}}^{(0)}_{*,i} = \epsilon_i^{\rm t}\, C_q\, C_{\rm in}\, 1 = \epsilon_i^{\rm t}\, (1,2,\ldots,K+1)^{\rm t} = i+1 = n_i \ ,$$

where ϵ_i is the i th standard unit vector, with rows labeled from 0 to K. ∎

As a consequence, we obtain an (L_0-graded) isomorphism between the space $\widehat{\mathcal{P}}_*$ of all (finite) paths over the extended Bratteli diagram $\widehat{\mathcal{B}}_q$ and the total Hilbert space $\mathcal{H}^{\rm tot}_{(q)}$,

$$\widehat{\mathcal{P}}_* \cong \mathcal{H}^{\rm tot}_{(q)} \ : \tag{3.13}$$

In order to show this, we first label the paths in $\widehat{\mathcal{P}}^{(0)}_{*,i}$ by ν running from 1 to n_i; then we identify a state in the ν th copy of $\mathcal{H}_i$ with a path on $\widehat{\mathcal{B}}_q$ which reaches node i on the 0 th floor along the path $|\nu\rangle$, and continues towards infinity according to the path space representations of each $\mathcal{H}_i$ introduced in section 2.2.

With a different choice of multiplicities n_i, it would still be possible to find an extended Bratteli diagram whose associated path space is isomorphic to $\mathcal{H}^{\rm tot}$, but it could be very different in shape from the one determined by the graph $\mathcal{G}_q$. In other words, the path field algebra associated to the extended Bratteli diagram, see below, would no longer be of the same stable isomorphism type as the $\mathcal{A}_i$.

Let us, for one more time, refer to the ideas of section 3.2 where the multiplicities n_i were interpreted as the dimensions of special subspaces $\mathcal{H}^{\rm s}_i \subset \mathcal{H}_i$. For degenerate Virasoro models, we could simply choose

$$\mathcal{H}^{\rm s}_i = \big\{\, L^m_{-1}|h_i\rangle \,\big|\, m \geq 0,\ L^m_{-1}|h_i\rangle \text{ linearly independent of } L_n\mathcal{H}_i \text{ for all } n \leq -2 \,\big\}_{\mathbb{C}} \ .$$

With this information, we may derive that $\widehat{\mathcal{P}}_* \cong \mathcal{H}^{\rm tot}_{(q)}$ even without knowledge of the values of n_i: Recall that by Proposition 2.2 and with the notations of (2.22) we have

$$\mathcal{H}^{\rm s}_i \ \cong\ \bigoplus_{l=0}^{K} P^{(1)}_{i,l} \ \cong\ \bigoplus_{l=0}^{K} P^{(1)}_{l,i} \ ;$$

the second relation holds since the graphs $\mathcal{G}_q$ are unoriented. But the last decomposition just describes the space of paths on the two extra floors of $\widehat{\mathcal{B}}_q$, ending at node i on floor 0. Since from floor zero on, the extended diagram is as the diagrams $\mathcal{B}_{q,i}$, its associated path space $\widehat{\mathcal{P}}_*$ is simply $\bigoplus_i \mathcal{H}_i \otimes \mathcal{H}_i^{\mathrm{s}}$, no matter what the dimensions of the small spaces are. By construction, this space is our total space of states.

Given the path representation of the total Hilbert space, the action of the quantum symmetry algebra is implemented in a straightforward way. For each fixed q, we denote by $\mathcal{F}_{*,i}^{(n)}$ the string algebra over $\widehat{\mathcal{B}}_q$ generated by pairs of paths joining node $*$ on floor -2 to node i on floor n. We set $\mathcal{F}_*^{(n)} := \bigoplus_i \mathcal{F}_{*,i}^{(n)}$, and we enumerate the paths $|\nu\rangle \in \widehat{\mathcal{P}}_{*,i}^{(0)}$ from $\nu = 0, \dots, n_i$ as before.

Definition 3.6 The quantum symmetry algebra $\mathsf{g} = \bigoplus_i M_{n_i}(\mathbb{C})$ acts as an associative algebra on the total Hilbert space $\mathcal{H}^{\mathrm{tot}}$ by the representation U which is defined on matrix units $E_{\mu\nu}^i \in M_{n_i}(\mathbb{C})$ as

$$U\big(E_{\mu\nu}^i\big) = |\mu\rangle\langle\nu| \in \mathcal{F}_{*,i}^{(0)} \ ;$$

in other words, $U(\mathsf{g}) = \mathcal{F}_*^{(0)}$.

This natural action of g on $\mathcal{H}^{\mathrm{tot}}$ is unital and faithful. The operators $U(a)$, $a \in \mathsf{g}$, flip only the first two edges of a path in $\widehat{\mathcal{P}}_*$, leaving already the node on the 0 th floor fixed. On the highest weight state $|h_i^\mu\rangle$ of the μ th copy of $\mathcal{H}_i$, $\mu = 1, \dots, n_i$, they act as

$$U(a)|h_i^\mu\rangle = \sum_{\nu=1}^{n_i} (a_i)_{\mu\nu}\, |h_i^\nu\rangle \ ;$$

the complex numbers $(a_i)_{\mu\nu}$ are the matrix entries of the i th factor of $a \in \mathsf{g}$. As a consequence, the vacuum is "invariant" under g in the sense that it transforms in the trivial representation given by the co-unit. (This fact reminds us that g generalizes the group algebra rather than the Lie algebra of a global gauge group.) In particular, we conclude that the g-action has no "observable effect":

Proposition 3.7 Let $\mathcal{F}$ denote the path algebra generated by all finite strings over $\widehat{\mathcal{B}}_q$. The commutant of $U(\mathsf{g})$ in $\mathcal{F}$ is canonically isomorphic to the observable algebra,

$$U(\mathsf{g})' \cap \mathcal{F} \cong \mathcal{A} \ .$$

PROOF: The claim is more or less obvious, and the proof will be given mainly in order to set up a matrix notation for the elements of $\mathcal{F}$ which will be useful in later computations. The identification of elementary strings with matrix units has already been used for finite floor sub-algebras; now, it will be applied to make the form of field algebra elements explicit up to a certain floor: For fixed $q = 2K + 3$, let $N = n_0 + n_1 + \ldots + n_K = \frac{(K+1)(K+2)}{2}$ be the total number of paths on $\widehat{\mathcal{B}}_q$ from $*$ to all the nodes on the zeroth floor. Then we can write any element $F \in \mathcal{F}$ as matrix

$$F = \big(F_{rs}\big)_{r,s=1}^{N} \tag{3.14}$$

where F_{rs} is a linear combination of strings starting at $*$, taking the routes $|r\rangle\langle s|$ down to the 0 th floor, and then continuing arbitrarily (such that the ends eventually meet). In this notation, the operators in $U(\mathsf{g})$ have the block-diagonal form

$$U(a) \in \mathrm{bl\,diag}\left(\mathbb{C}\cdot\mathbf{1}_{\mathcal{A}_0}, M_2(\mathbb{C}\cdot\mathbf{1}_{\mathcal{A}_1}), \ldots, M_{K+1}(\mathbb{C}\cdot\mathbf{1}_{\mathcal{A}_K})\right)$$

for all $a \in \mathsf{g}$, where $\mathbf{1}_{\mathcal{A}_i}$ is the unit of the path algebra $\mathcal{A}_i$. The commutant of $U(\mathsf{g})$ in $\mathcal{F}$ is then simply given by

$$U(\mathsf{g})' = \mathrm{bl\,diag}\left(\mathcal{A}_0, D_2(\mathcal{A}_1), \ldots, D_{K+1}(\mathcal{A}_k)\right) \tag{3.15}$$

where $D_i\,:\,\mathcal{A} \longrightarrow \mathcal{A}^{\oplus i},\ a \longmapsto a\oplus\cdots\oplus a$ is the diagonal embedding; obviously, $U(\mathsf{g})'$ is canonically isomorphic to $\mathcal{A}$. ∎

In view of this proposition, we will call $\mathcal{F}$ the (path representation of the) *field algebra* of the $c(2,q)$-model. Of course, like our path observable algebra, $\mathcal{F}$ is a global object. As an AF-algebra, $\mathcal{F}$ once more belongs to the same stable isomorphism class as all the path algebras associated to the path representations $\mathcal{H}_i$, which is clear from the form of the Bratteli diagram $\widehat{\mathcal{B}}_q$. Unlike the global observable algebra $\mathcal{A}$, however, $\mathcal{F}$ is a simple algebra. Note also that $\mathcal{F} \neq U(\mathsf{g}) \vee U(\mathsf{g})'$, but the centers of the QSA and the observable algebra $\mathcal{A}$ coincide on the total Hilbert space,

$$Z(\mathcal{A}) = Z(U(\mathsf{g}))\ ,$$

in agreement with the general theory. Compared to the huge field algebras constructed elsewhere, our path field algebra $\mathcal{F}$ envelops the path observable algebra $\mathcal{A}$ rather tightly.

4. Covariant field multiplets

This section contains the main step towards global amplimorphisms of the $c(2,q)$ minimal models, namely the construction of covariant field multiplets inside the field algebra $\mathcal{F}$. Once these are known, the amplimorphisms follow immediately. In order to arrive at the multiplets, the QSA is first endowed with a co-product Δ reproducing the fusion rules of the $c(2,q)$ model – or rather with an equivalent collection of amplimorphisms ν_i, $i = 0,\ldots,K$, of g. With the help of these amplimorphisms, one can formulate equations on elements of $\mathcal{F}$ which are to form covariant multiplets. In our cases, the special properties of the underlying path spaces allow for a natural solution of the covariance conditions.

4.1 Amplimorphisms of the QSA

For the moment, we treat g as an abstract matrix algebra $\mathsf{g} = \bigoplus_{i=0}^{K} M_{n_i}(\mathbb{C})$. The very first condition g has to meet in order to become the QSA of a CFT is existence of a co-product $\Delta\,:\,\mathsf{g} \longrightarrow \mathsf{g}\otimes\mathsf{g}$ which reproduces the fusion rules of the CFT, see section 3.1. This means that the (two-floor) Bratteli diagram of the algebra homomorphism $\Delta\,:\,\mathsf{g} \longrightarrow \mathsf{g}\otimes\mathsf{g}$ is

fixed, but there is the freedom of "twisting" the co-product by inner automorphisms of $\mathsf{g} \otimes \mathsf{g}$.

Given $\Delta : \mathsf{g} \longrightarrow \mathsf{g} \otimes \mathsf{g}$, we can use the minimal central projections e_i, $i = 0, \ldots, K$, to introduce *amplimorphisms* [62]

$$\nu_i : \begin{cases} \mathsf{g} \longrightarrow M_{n_i}(\mathsf{g}) \\ a \longmapsto \nu_i(a) := (\mathbf{1} \otimes e_i)\Delta(a) \end{cases}$$

of g; here, the isomorphism $M_{n_i}(\mathsf{g}) \cong \mathsf{g} \otimes M_{n_i}(\mathbb{C})$ has tacitly been applied.

Vice versa, a collection of amplimorphisms $\nu_i : \mathsf{g} \longrightarrow M_{n_i}(\mathsf{g})$ for $i = 0, \ldots, K$, of the above type defines a co-product

$$\Delta_{\{\nu\}} : \begin{cases} \mathsf{g} \longrightarrow \mathsf{g} \otimes \mathsf{g} \\ a \longmapsto \Delta(a) := \sum_{i=0}^{K} \nu_i(a) \end{cases}$$

– up to inner automorphism, since again an explicit isomorphism from $M_{n_i}(\mathsf{g})$ to $\mathsf{g} \otimes \mathsf{g}_i$, $g_i = e_i \cdot \mathsf{g}$, has to be chosen. The invariant information contained in Δ and in the collection of g-amplimorphisms is, however, the same, and we will see that the objects we aim at, namely the amplimorphisms of the observable algebra, are independent of twists of Δ.

To work with amplimorphisms of the semi-simple QSA instead of the co-product was proposed by Szlachanyi and Vecsernyes, and the idea has been applied to G-spin chains and to the Ising model [60, 62]. Amplimorphisms seem to be better adapted to the DHR framework, and, for some purposes, are easier to handle in practice.

Our task is, given the $c(2,q)$-fusion rules N_{ij}^k, find amplimorphisms ν_i of g whose Bratteli diagrams contain N_{ij}^k lines from g_k to $M_{n_i}(\mathsf{g}_j)$. This is fairly easy to do, in fact, the main difficulties in writing down the ν_i are of notational type. Fix a superselection sector $i \in I_q$, and choose, for each $j \in I_q$, an enumeration

$$(ij|1), (ij|2), \ldots, (ij|m_{ij}) \tag{4.1}$$

of the fusion results $|i-j|, \{|i-j|+2\}, \ldots, i+j$ in the decomposition of $\phi_i \times \phi_j$, see eqs. (2.4-6); recall that $m_{ij} = \min(i,j)+1$. Using this notation, and the decomposition of g and its elements with the help of the minimal central projections, $a_i := a \cdot e_i$, we define the auxiliary map

$$\widetilde{\nu_i} : \begin{cases} \mathsf{g} \longrightarrow M_{n_i}(\mathsf{g}_0) \oplus M_{n_i}(\mathsf{g}_1) \oplus \cdots \oplus M_{n_i}(\mathsf{g}_K) \\ a \longmapsto \widetilde{\nu_i}(a) \end{cases} \tag{4.2}$$

by

$$\widetilde{\nu_i}(a) := a_i \oplus \operatorname{bl\,diag}\big(a_{(i1|1)}, \ldots, a_{(i1|m_{i1})}, 0_{d_{i1}}\big) \oplus \cdots \oplus \operatorname{bl\,diag}\big(a_{(iK|1)}, \ldots, a_{(iK|m_{iK})}, 0_{d_{iK}}\big) \ ;$$

$0_{d_{ij}}$ denotes a square matrix of size

$$d_{ij} := n_i n_j - \sum_{l=1}^{m_{ij}} n_{(ij|l)} \tag{4.3}$$

with all entries equal to zero. The d_{ij} are just the defects in the inequalities (3.2). In the terminology of section 3.2, d_{ij} is the dimensions of the "spurious space" in the fusion of the representations i and j.

The target space of $\widetilde{\nu_i}$ is isomorphic to $M_{n_i}(\mathsf{g})$, and we arrive at a true amplimorphism of the QSA if we choose a specific isomorphism. Since both $\bigoplus_l M_{n_i}(\mathsf{g}_l)$ and $M_{n_i}(\mathsf{g})$ embed canonically into $M_{n_i N}(\mathbb{C})$, $N = \sum_l n_l$, we can simply use a permutation matrix $P_\pi \in M_{n_i N}(\mathbb{C})$ in order to "rearrange" the matrix $\widetilde{\nu_i}(a) \in M_{n_i N}(\mathbb{C})$. The permutation $\pi \in S_{n_i N}$ is obtained from comparing the following two labelings of the standard basis ϵ_s, $s = 1, \ldots, n_i N$ of $\mathbb{C}^{n_i N}$:

Labeling I is in terms of triples $(n_j, \sigma; \alpha)$ with $j = 0, \ldots, K$, $\sigma = 1, \ldots, n_j$, $\alpha = 1, \ldots, n_i$. Here, $\epsilon_{(n_j,\sigma;\alpha)} = \epsilon_s$ with $s = n_i(n_0 + \ldots + n_{j-1}) + (\alpha - 1)n_j + \sigma$.

Labeling II is in terms of triples $(\alpha; n_j, \sigma)$ with the same range of indices. Here, $\epsilon_{(\alpha;n_j,\sigma)} = \epsilon_s$ with $s = (\alpha - 1)N + (n_0 + \ldots + n_{j-1}) + \sigma$.

The triples $(n_j, \sigma; \alpha)$ of labeling I provide a natural row-column ordering for the sub-algebra $\bigoplus_l M_{n_i}(\mathsf{g}_l)$ of $M_{n_i N}(\mathbb{C})$, whereas the triples $(\alpha; n_j, \sigma)$ of labeling II are appropriate when we consider the sub-algebra $M_{n_i}(\mathsf{g})$. Accordingly, let $\pi \in S_{n_i N}$ be the permutation $s \longmapsto \pi(s)$ with

$$\pi(s) = (\alpha; n_j, \sigma) \quad \text{for} \quad s = (n_j, \sigma; \alpha) \tag{4.4}$$

in an obvious notation, and let P_π be the $n_i N \times n_i N$ matrix whose sth column is the standard unit vector $\epsilon_{\pi(s)}$.

Proposition 4.1 For all $i \in I_q$, the map

$$\nu_i \ : \ \begin{cases} \ \mathsf{g} \longrightarrow M_{n_i}(\mathsf{g}) \\ \ a \longmapsto P_\pi^{\mathrm{t}} \, \widetilde{\nu_i}(a) \, P_\pi \ , \end{cases}$$

with P_π determined by (4.4) as above, is an amplimorphism of the QSA g such that any associated co-product $\Delta_{\{\nu\}}$ reproduces the $c(2,q)$-fusion rules.

Proof: First, we have to show that $P_\pi^{\mathrm{t}} \, \widetilde{\nu_i}(a) \, P_\pi \in M_{n_i N}(\mathbb{C})$ is contained in the sub-algebra $M_{n_i}(\mathsf{g})$ for all $a \in \mathsf{g}$. This is true since $\widetilde{\nu_i}(a)$ is an element of the block-diagonal sub-algebra $M_{n_i}(\mathsf{g}_0) \oplus \cdots \oplus M_{n_i}(\mathsf{g}_K)$ of $M_{n_i N}(\mathbb{C})$, which means, in terms of labeling I, that $\big(\widetilde{\nu_i}(a)\big)_{(n_j,\sigma;\alpha)(n_k,\tau;\beta)} \neq 0$ only if $j = k$; the same applies for $\big(\nu_i(a)\big)_{(\alpha;n_j,\sigma)(\beta;n_k,\tau)}$, only now the counting is wrt labeling II, and $\nu_i(a) \in M_{n_i}(\mathsf{g})$ follows. An explicit formula for the matrix elements of the amplimorphism ν_i is

$$\big(\nu_i(a)\big)_{(\alpha;n_j,\sigma)(\beta;n_k,\tau)} = \delta_{jk} \sum_{l=1}^{m_{ij}} \big(a_{(ij|l)}\big)_{f_l(\alpha,\sigma),f_l(\beta,\tau)} \tag{4.5a}$$

with

$$f_l(\alpha,\sigma) = \sigma + (\alpha-1)n_j - (n_{(ij|1)} + \ldots + n_{(ij|l-1)}) \ , \tag{4.5b}$$

and we use the convention that the matrix element $(a_l)_{\rho\sigma}$ vanishes unless $1 \le \rho, \sigma \le n_l$. Thus, at most one of the terms in the sum of eq. (4.5a) gives a non-zero contribution. The assertion on the Bratteli diagram associated to the amplimorphism ν_i (and to $\Delta_{\{\nu\}}$) is enforced by the very construction of the auxiliary maps $\widetilde{\nu}_i$, which have been chosen as the simplest possible realization of the Bratteli diagram dictated by the fusion rules. Both in the definition of $\widetilde{\nu}_i$ and of ν_i one could introduce additional twists by unitaries. ∎

Note that the amplimorphisms ν_i are *-homomorphisms with respect to the canonical *-operation on complex matrix algebras and, more importantly, that they are all non-unital – except for $\nu_0 = \mathrm{id}$. This non-unitality may be traced back to the non-zero defects d_{ij} of eq. (4.3), but the rank of $\nu_i(1)$ does not depend on the specifically simple choice of amplimorphisms ν_i. Likewise, the following important statement is one on the inner isomorphism class of the g-amplimorphism:

Proposition 4.2 Upon composition, the amplimorphisms ν_i of g realize the fusion rules, i.e. for each pair $i,j \in I_q$, there exists a unitary $U_{ij} \in M_{n_i n_j}(\mathsf{g})$ such that

$$(\nu_i \circ \nu_j)(a) = U_{ij}^* \,\Big(\bigoplus_{k\in I_q} \nu_k(a)^{\oplus N_{ij}^k}\Big)\, U_{ij}$$

for all $a \in \mathsf{g}$ and the fusion rules given in section 2.1. In particular, $\nu_i \circ \nu_j$ and $\nu_j \circ \nu_i$ are unitarily equivalent.

Proof: Recall that an algebra homomorphism $\psi \ : \ A \longrightarrow B$ is extended to an algebra homomorphism $\psi \ : \ M_n(A) \longrightarrow M_n(B)$ of the amplifications by setting $\psi\big((a_{ij})_{i,j=1}^n\big) = \big(\psi(a_{ij})\big)_{i,j=1}^n$. Thus, the lhs is a map from g to $M_{n_i}(M_{n_j}(\mathsf{g}))$, the rhs can be regarded as an element of $M_{n_i n_j}(\mathsf{g})$ because of the basic inequality $n_i n_j \ge \sum_k N_{ij}^k n_k$. The existence of the unitaries U_{ij} is clear since in the Bratteli diagrams associated to $\nu_i \circ \nu_j$ and $\bigoplus_k \nu_k^{\oplus N_{ij}^k}$ there are $\sum_k N_{im}^k N_{jk}^l$ resp. $\sum_k N_{ij}^k N_{km}^l$ lines from the factor g_l to the factor $M_{n_i n_j}(\mathsf{g}_m)$: The diagrams coincide. Note that with our simple realization of the amplimorphisms, the unitaries U_{ij} are in fact permutation matrices. ∎

Let us illustrate the notions of this section in the simplest model of our series, namely the CFT describing the Lee-Yang edge singularity of the Ising model. To the $c(2,5)$ minimal model, we associate the QSA

$$\mathsf{g}_{(5)} = \mathbb{C} \oplus M_2(\mathbb{C}) \ .$$

(In the language of small spaces from subsection 3.2, $\dim \mathcal{H}_0^{\mathrm{s}} = 1$ for the vacuum representation is a general fact, and $\dim \mathcal{H}_1^{\mathrm{s}} = 2$ follows from the presence of the null-vector $(L_{-1}^2 - \frac{2}{5}L_{-2})|h_1\rangle = 0$ in the irreducible module $\mathcal{H}_1$ of the $c(2,5)$ theory.)

The total Hilbert space $\mathcal{H}_{(5)}^{\mathrm{tot}} = \mathcal{H}_0 \oplus (\mathcal{H}_1 \otimes \mathbb{C}^2)$ is identified with the path space over the Bratteli diagram $\widehat{\mathcal{B}}_5$, see Figure 2, and the global (path) field algebra is the string

algebra over $\widehat{\mathcal{B}}_5$. The representation of $\mathsf{g}_{(5)}$ on $\mathcal{H}^{\mathrm{tot}}_{(5)}$ is implemented in a straightforward way following Dfinition 3.6.

The amplimorphisms of $\mathsf{g}_{(5)}$ are also obtained easily: $\nu_0 = \mathrm{id}$ is trivial, and the fusion rule $\phi_1 \times \phi_1 = \phi_0 + \phi_1$ with this enumeration of fusion results, i.e. $(11|1) = 0,\ (11,2) = 1$, yields

$$\widetilde{\nu_1}(a) = \begin{pmatrix} a_1^{11} & a_1^{12} \\ a_1^{21} & a_1^{22} \end{pmatrix} \oplus \begin{pmatrix} a_0 & 0 & 0 & 0 \\ 0 & a_1^{11} & a_1^{12} & 0 \\ 0 & a_1^{21} & a_1^{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

for the auxiliary morphism $\widetilde{\nu_1} : \mathsf{g} \longrightarrow M_2(\mathsf{g}_0) \oplus M_2(\mathsf{g}_1)$, where $a = a_0 \oplus \begin{pmatrix} a_1^{11} & a_1^{12} \\ a_1^{21} & a_1^{22} \end{pmatrix} \in \mathsf{g}$. In the $c(2,5)$-model, the defect is $d_{11} = 1$, explaining the zero in the lower right corner of the matrix $\widetilde{\nu_1}(a)$.
The permutation P_π which has to be applied in passing to an amplimorphism $\nu_1 : \mathsf{g} \longrightarrow M_2(\mathsf{g})$ amounts to slicing the matrices in $\widetilde{\nu_1}(a)$ into quarters and collecting them together as

$$\nu_1(a) = \begin{pmatrix} a_1^{11} \oplus \begin{pmatrix} a_0 & 0 \\ 0 & a_1^{11} \end{pmatrix} & a_1^{12} \oplus \begin{pmatrix} 0 & 0 \\ a_1^{12} & 0 \end{pmatrix} \\ & \\ a_1^{21} \oplus \begin{pmatrix} 0 & a_1^{21} \\ 0 & 0 \end{pmatrix} & a_1^{22} \oplus \begin{pmatrix} a_1^{22} & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} . \tag{4.6}$$

It is straightforward to verify that $(\nu_1 \circ \nu_1)(a)$ and $\nu_0(a) \oplus \nu_1(a)$ are equal up to simultaneous permutation of rows and columns.

4.2 Construction of covariant field multiplets

In this subsection, we will construct multiplets of "charged fields" which transform covariantly under the QSA action, and which can be used to define (global) amplimorphisms of the path observable algebra $\mathcal{A}$.
The charged fields associated with a sector $i \in I_q$ are operators $F_i = \big((F_i)_{\alpha\beta}\big)_{\alpha,\beta=1}^{n_i} \in M_{n_i}(\mathcal{F})$ in the n_i'th amplification of the path field algebra $\mathcal{F}$, and they are subject to the conditions

$$a \cdot \big(F_i\big)_{\alpha\beta} = \big(F_i \cdot \nu_i(a)\big)_{\alpha\beta} \qquad \text{for all } a \in \mathsf{g} \ , \tag{4.7}$$

$$F_i^* F_i = \nu_i(1) \ . \tag{4.8}$$

In both equations, we have identified $a \in \mathsf{g}$ with its image $U(a)$ on $\mathcal{H}^{\mathrm{tot}}$. Eq. (4.7) simply expresses the g-covariance of the charged multiplets, written in terms of g-amplimorphisms, cf. [62], rather than the co-product as in [58]. The second relation, where "1" of course is the unit of the QSA g, can be viewed as a completeness condition; however, since the g-amplimorphisms ν_i are i.g. non-unital, the field multiplets F_i are merely partial isometries in $M_{n_i}(\mathcal{F})$.

Finding explicit solutions of (4.7) and (4.8) is the only slightly technical part of the constructions presented in this paper. The matrix notation of eq. (3.14) for the field operators $(F_i)_{\alpha\beta}$ will prove useful in this process, only now we will group the indices according to "labeling II" of the last subsection. Thus, we write

$$\big(F_i\big)_{\alpha\beta} = \big((F_i)_{(\alpha;n_k,\sigma)(\beta;n_j,\rho)}\big)_{k,\sigma;j,\rho} \tag{4.9}$$

with $j,k \in I_q$, $\rho = 1,\dots,n_j$, $\sigma = 1,\dots,n_k$. Again, this notation is to make the first two floors of a string in $\mathcal{F}$ "visible": The indices $(n_j,\rho)(n_k,\sigma)$ of the $(F_i)_{\alpha\beta}$-entry indicate that it is a (linear combination of) string(s) of the form $|p^k_\infty \circ \sigma^*_k\rangle\langle p^j_\infty \circ \rho^*_j|$ where $|\rho^*_j\rangle$ is the ρ'th path from $*$ to the node j on the 0'th floor, and $|p^j_\infty\rangle \in \mathcal{P}_j$ runs from j on to infinity (with the usual tail condition); the symbol $\circ$ denotes concatenation of the two pieces. We will call such an element of $\mathcal{F}$ with indices $(n_k,\sigma)(n_j,\rho)$ a *k-j-string* for short (though it is actually only a "half-open" string).
Since we can identify states in the irreducible representations $\mathcal{P}_j$ with *j-paths* on the extended Bratteli diagram (in obvious terminology), we find that the field operators carry one representation into another. More precisely, the matrix element (4.9) maps the (ρ'th copy of the) space $\mathcal{H}_j$ into the (σ'th copy of the) space $\mathcal{H}_k$.

In the following, we will first deal with the covariance condition (4.7), which turns out to determine only the "coarse structure" of the operators $\big(F_i\big)_{\alpha\beta}$ – i.e. to determine that some of the matrix elements (4.9) have to vanish, and some have to coincide. Afterwards, the completeness relation (4.8) will place non-trivial constraints on the non-vanishing entries in $\big(F_i\big)_{\alpha\beta}$, and it is this step where the combinatorial structure of the path spaces becomes essential in solving the constraints.

Proposition 4.3 The field multiplet $\big(F_i\big)_{\alpha\beta}$, $\alpha,\beta = 1,\dots,n_i$, transforms covariantly under the g-action if and only if the matrix elements (4.9) have the following form:

$$(F_i)_{(\alpha;n_k,\sigma)(\beta;n_j,\rho)} = C^k_{ij;\,\alpha}\ \sum_{l=1}^{m_{ij}} \delta_{(ij|l),k}\ \delta_{\sigma+n_{(ij|1)}+\ldots+n_{(ij|l-1)}\,,\,\rho+(\beta-1)n_j}$$

Here, $C^k_{ij;\,\alpha}$ denotes some k-j-string on the Bratteli diagram $\widehat{\mathcal{B}}_q$. In particular, the matrix elements vanish unless ϕ_k occurs in the fusion decomposition of $\phi_i \times \phi_j$.

Proof: One advantage of the explicit matrix notation for field algebra elements is that the QSA-operators $a \in \mathsf{g}$ commute with the entries $C^k_{ij;\,\alpha}$ of $\big(F_i\big)_{\alpha\beta}$, since $a \equiv U(a)$ acts non-trivially only between $*$ and the 0'th floor of $\widehat{\mathcal{B}}_q$, whereas the $C^k_{ij;\,\alpha}$ are made up from half-open string starting on the 0'th floor. Therefore, we can simply treat $C^k_{ij;\,\alpha}$ as numerical coefficients, for the time being.

With F_i as above, let us calculate the lhs of eq. (4.7):

$$\begin{aligned}
\big(a\cdot(F_i)_{\alpha\beta}\big)_{(n_m,\tau)(n_j,\rho)} &= \sum_{(n_k,\sigma)} a_{(n_m,\tau)(n_k,\sigma)}\ (F_i)_{(\alpha;n_k,\sigma)(\beta;n_j,\rho)} \\
&= \sum_{(n_k,\sigma)} \delta_{m,k}\ (a_m)_{\tau,\sigma}\ C^k_{ij;\,\alpha} \sum_{l=1}^{m_{ij}} \delta_{(ij|l),k}\ \delta_{\sigma+n_{(ij|1)}+\cdots+n_{(ij|l-1)}\,,\,\rho+(\beta-1)n_j} \\
&= C^m_{ij;\,\alpha} \sum_{l=1}^{m_{ij}} \delta_{(ij|l),m}\ (a_m)_{\tau,f_l(\beta,\rho)}
\end{aligned}$$

with $f_l(\beta,\rho) = \rho + (\beta-1)n_j - n_{(ij|1)} - \ldots - n_{(ij|l-1)}$ from (4.5b). For the rhs of (4.7), we obtain

$$\begin{aligned}
\big(F_i\nu_i(a)\big)_{(\alpha;n_m,\tau)(\beta;n_j,\rho)} = \sum_{(\gamma;n_k,\sigma)} & C^m_{ik;\,\alpha} \sum_{l=1}^{m_{ij}} \delta_{(ij|l),k}\ \delta_{\tau+n_{(ik|1)}+\ldots+n_{(ik|l-1)}\,,\,\sigma+(\gamma-1)n_k} \\
& \times\ \delta_{k,j} \sum_{l'=1}^{m_{ik}} \big(a_{(ik|l')}\big)_{f_{l'}(\gamma,\sigma),f_{l'}(\beta,\rho)} \\
= C^m_{ij;\,\alpha} \sum_{l,l'=1}^{m_{ij}} \delta_{(ij|l),m}\ & \big(a_{(ik|l')}\big)_{g_{l,l'}(\tau),f_{l'}(\beta,\rho)}\ ;
\end{aligned}$$

here, we have introduced the shorthand $g_{l,l'}(\tau) = \tau + n_{(ik|1)} + \ldots + n_{(ik|l-1)} - n_{(ik|1)} - \ldots - n_{(ik|l'-1)}$ for the first index. Now recall our convention that $(a_m)_{\rho,\sigma} = 0$ unless $1 \le \rho,\sigma \le n_m$, and also that we have once and for all fixed an enumeration of the fusion results $(ij|1),\ldots,(ij|m_{ij})$ in eq. (4.1). Put together, this means that the last expression contains an implicit Kronecker symbol setting $l = l'$ and, therefore, it agrees with the lhs of (4.7) calculated before.
Given the formula (4.5) for the g-amplimorphisms, one can also show that the multiplets F_i must have the form given in the proposition in order to solve (4.7). One may e.g. insert matrix units for $a \in \mathsf{g}$ to achieve complete decoupling of all equations, and the only difficulty is to keep track of the indices. Since this is slightly tedious, and since later we will not aim at that the most general solution for the coefficients $C^k_{ij;\,\alpha}$, we omit details of the "only if" part of the proof. ∎

Let us now turn to the completeness relation. We first prove the following intermediate result, which does not yet involve the special structure of our path spaces.

Proposition 4.4 The field multiplets F_i solve the completeness relation (4.8) if and only if the coefficients $C^k_{ij;\,\alpha}$ of Proposition 4.3 satisfy

$$\sum_{\alpha=1}^{n_i} \big(C^k_{im;\,\alpha}\big)^* C^k_{ij;\,\alpha} = \delta_{N^k_{ij},1}\,\delta_{m,j}\ \mathbf{1}_{\mathcal{P}_j}\ , \tag{4.10}$$

where $\mathbf{1}_{\mathcal{P}_j}$ is the identity operator on the path space $\mathcal{P}_j$ of section 2, viewed as subspace of $\widehat{\mathcal{P}}_{(q)}$.

Proof: Assume that (4.10) holds, and insert the formula for F_i from Proposition 4.3 into the lhs of (4.8); this yields

$$\begin{aligned}
\big(F_i^* F_i\big)_{(\alpha;n_m,\tau)(\beta;n_j,\rho)} &= \sum_{(\gamma;n_k,\sigma)} \big((F_i)_{(\gamma;n_k,\sigma)(\alpha;n_m,\tau)}\big)^* (F_i)_{(\gamma;n_k,\sigma)(\beta;n_j,\rho)} \\
&= \sum_{(\gamma;n_k,\sigma)} \big(C^k_{im;\gamma}\big)^* C^k_{ij;\gamma} \sum_{l'=1}^{m_{im}} \sum_{l=1}^{m_{ij}} \delta_{(im|l'),k}\, \delta_{(ij|l),k} \\
&\qquad \times \delta_{\sigma+n_{(im|1)}+\ldots+n_{(im|l'-1)}\,,\,\tau+(\alpha-1)n_m}\, \delta_{\sigma+n_{(ij|1)}+\ldots+n_{(ij|l-1)}\,,\,\rho+(\beta-1)n_j} \\
&= \mathbf{1}_{\mathcal{P}_j} \delta_{m,j} \sum_{(n_k,\sigma)} \sum_{l=1}^{m_{ij}} \delta_{(ij|l),k}\ \delta_{\tau+(\alpha-1)n_j\,,\,\rho+(\beta-1)n_j}\ \delta_{\sigma+n_{(ij|1)}+\ldots+n_{(ij|l-1)}\,,\,\tau+(\alpha-1)n_j}\ .
\end{aligned}$$

We have again used the fact that the enumeration of the fusion results was fixed, which enforces $l = l'$ above. Since the range of both τ and ρ is $1,\ldots,n_j$, the last but one Kronecker symbol implies $\tau = \rho$ *and* $\alpha = \beta$. Thus, $F_i^* F_i$ is diagonal,

$$\big(F_i^* F_i\big)_{(\alpha;n_m,\tau)(\beta;n_j,\rho)} = \delta_{\alpha,\beta}\, \delta_{m,j}\, \delta_{\tau,\rho}\, \Theta_{ij}(\alpha,\tau)\ \mathbf{1}_{\mathcal{P}_j}$$

with a "cutoff factor"

$$\Theta_{ij}(\alpha,\tau) := \begin{cases} 1 & \text{if } \sum_k N_{ij}^k n_k \geq \tau + (\alpha-1)n_j\,, \\ 0 & \text{otherwise.} \end{cases}$$

The rhs of (4.8) is only a special case of (4.5),

$$\big(\nu_i(1)\big)_{(\alpha;n_m,\tau)(\beta;n_j,\rho)} = \mathbf{1}_{\mathcal{P}_j}\ \delta_{m,j} \sum_{l=1}^{m_{ij}} \big(1_{(ij|l)}\big)_{f_l(\alpha,\tau),f_l(\beta,\sigma)}\ ,$$

which we have multiplied by the unit operator on $\mathcal{P}_j$ as we are actually working with $U(\nu_i(1))$ acting on $\widehat{\mathcal{P}}_{(q)}$. Since $1 \in \mathfrak{g}$ is a diagonal matrix, the elements above vanish unless $f_l(\alpha,\tau) = f_l(\beta,\sigma)$, i.e. unless $\tau = \rho$ *and* $\alpha = \beta$, and a closer look at the "defect" of the non-unital amplimorphism ν_i shows that the cutoff factor $\Theta_{ij}(\alpha,\tau)$ appears as well.
Proving the reverse direction is easier after having a closer look at the structure of the matrix elements of F_i in Proposition 4.3: Note that the Kronecker symbols are independent of α and that, for fixed α, there is at most one non-zero entry in each column $(\beta;n_j,\rho)$. Thus, the entries in $F_i^* F_i$ are precisely of the form $\sum_\alpha \big(C^k_{im;\alpha}\big)^* C^k_{ij;\alpha}$, and condition (4.10) follows from the matrix structure of $\nu_i(1)$. ∎

The actual task is to construct (half-open) strings $C^k_{ij;\gamma}$ which satisfy the relations (4.10). This will be done with the help of embeddings of path spaces, mapping elementary paths

to elementary paths. First, we need a combinatorial lemma comparing the sizes of certain path spaces.

Lemma 4.5 As in eq. (2.22), let $\mathcal{P}^{(2)}_{k,m}$ be the space of paths of length 2 on $\mathcal{G}_q$ which start from node k on the 0 th floor and end at node m on the 2 nd floor, $k,m \in I_q$. With the $c(2,q)$ fusion rules N_{ij}^k and the sector multiplicities $n_i = i+1$, the following estimate holds for all $i,k,m \in I_q$:

$$n_i \dim \mathcal{P}^{(2)}_{k,m} \geq \sum_{j\in I_q} N_{ij}^k \dim \mathcal{P}^{(2)}_{j,m}$$

Proof: As in the proof of Lemma 3.5, we express $\dim \mathcal{P}^{(2)}_{j,m}$ through the embedding matrix C_q and standard unit vectors; thus, the lhs is

$$n_i \dim \mathcal{P}^{(2)}_{k,m} = n_i\, \epsilon_m^{\mathrm{t}} C_q^2 \epsilon_k = n_i\, (N_K^2)_{mk} = n_i\, (N_0 + N_1 + \ldots + N_K)_{mk}\ .$$

Here, the fact that $\mathcal{G}_q$ is just the fusion graph of the minimal dimension field ϕ_K of the $c(2,q)$-model is very convenient: The last equality is the fusion rule $\phi_K \times \phi_K$. For the rhs, we compute in the same fashion that

$$\begin{aligned}\sum_k N_{ij}^k \dim \mathcal{P}^{(2)}_{j,m} &= \sum_k N_{ij}^k (N_K^2)_{mj} = (N_K^2 N_i)_{mk} \\ &= \big((i+1)(N_K + \ldots + N_i) + iN_{i-1} + \ldots + 2N_1 + N_0\big)_{mk}\ .\end{aligned}$$

The last step follows from applying the fusion rule $\phi_K \times \phi_i = \phi_K + \phi_{K-1} + \ldots + \phi_{K-i}$ twice, see section 2.1. Subtracting the rhs from the lhs, we obtain a matrix whose elements are all non-negative if and only if $n_i \geq i+1$. ∎

From the proof of this lemma, we learn as a by-product that our choice of sector multiplicities is indeed the *minimal* one such that the dimension estimate holds true – and, as a consequence, such that the construction of the $C^k_{ij:\,\alpha}$ to be given below is possible. This seems remarkable since up to now the special values $n_i = i+1$ were distinguished only on the general grounds of section 3.2, whereas the basic inequalities $n_i n_j \geq \sum_k N_{ij}^k n_k$ can in general be fulfilled with some of the multiplicities taken smaller than $i+1$.

Below, the following generalizations of Lemma 4.5 will be useful:

Corollary 4.6 For all $n \geq 2$, and for all $i,k,m \in I_q$, we have

$$n_i \dim \mathcal{P}^{(n)}_{k,m} \geq \sum_{j\in I_q} N_{ij}^k \dim \mathcal{P}^{(n)}_{j,m}\ .$$

Furthermore, if the sector indices are such that $k+m \geq K$, the dimension inequality also holds for $n=1$.

Proof: The first claim is true because all path spaces are based on the same fusion graph $\mathcal{G}_q$, so for $n \geq 2$ we obtain the dimensions of the path spaces $\mathcal{P}^{(n)}$ when applying C_q^{n-2} to

those of $\mathcal{P}^{(2)}$; this does not spoil the estimate in the previous lemma. The case $n = 1$ is only special as far as some of the spaces $\mathcal{P}^{(1)}_{k,m}$ are empty: These are precisely those with $k + m < K$, as follows from the form of $\mathcal{G}_q$. ∎

After these preparations, we are ready to construct strings $C^k_{ij;\,\alpha}$ with the property (4.10). Lemma 4.5 guarantees that for all $i, j, k, m \in I_q$ there exist injective homomorphisms

$$\iota^{k\ (2)}_{i,\,m} \ :\ \bigoplus_j \big(\mathcal{P}^{(2)}_{j,m}\big)^{\oplus N^k_{ij}} \longrightarrow \big(\mathcal{P}^{(2)}_{k,m}\big)^{\oplus n_i} \tag{4.11}$$

of path spaces of length 2, which leave the endpoints (here the node m) fixed. We arrange the injections in such a way that elementary paths are mapped to elementary paths. This allows us to extend $\iota^{(2)}$ to longer paths simply by requiring compatibility with concatenations c^l_m, see eq. (2.23): Given a collection of maps $\iota^{k\ (n)}_{i,\,l}$ for some $n \geq 2$, we define $\iota^{k\ (n+1)}_{i,\,m}$ by

$$\iota^{k\ (n+1)}_{i,\,m}\big(c^l_m(p)\big) = c^l_m\big(\iota^{k\ (n)}_{i,\,l}(p)\big) \tag{4.12}$$

for all elementary paths $|p\rangle \in \bigoplus_k \big(\mathcal{P}^{(n)}_{j,l}\big)^{\oplus N^k_{ij}}$. Moreover, the second part of Corollary 4.6 states that injections $\iota^{(1)}$ can already be defined for paths of length 1 at least in some cases. Among those are the first edge of the highest weight path (wrt the $L^{\mathcal{G}}_0$ action (2.21) in each $\mathcal{P}_i$, i.e. the edge $(i \to K)$; we choose $\iota^{k\,(1)}_i$ such as to map this edge to $(k \to K)$ – and whenever possible, we require already $\iota^{(2)}$ to be induced by $\iota^{(1)}$ according to (4.12).

Having chosen such a collection of injections $\iota^{k\ (2)}_{i,\,m}$ for all $i, k, m \in I_q$ – the choice involved is a finite one, to be discussed later – the compatibility with concatenation also ensures that we can take the inductive limit of the system $\big(\iota^{k\ (n)}_{i,\,m}\big)_n$, and we arrive at well-defined embeddings of infinite path spaces

$$\iota^k_i \ :\ \bigoplus_j \big(\mathcal{P}_j\big)^{\oplus N^k_{ij}} \longrightarrow \big(\mathcal{P}_k\big)^{\oplus n_i} \tag{4.13}$$

which by definition map elementary paths to elementary paths, preserve length and endpoint of every finite path and, in particular, map ground states to ground states.

We need two more (canonical) maps to be able to write down a formula for $C^k_{ij;\,\alpha}$. One is the projection from the n_i-fold direct sum of $\mathcal{P}_k$ onto the α'th factor, $\alpha = 1, \ldots, n_i$,

$$\mathrm{pr}_\alpha \ :\ \big(\mathcal{P}_k\big)^{\oplus n_i} \longrightarrow \mathcal{P}_k\ . \tag{4.14}$$

The other is the inclusion of $\mathcal{P}_j$ into the direct sum of path spaces occurring in the fusion of i and k – which, however, vanishes if $N^k_{ij} = 0$. We write

$$\varepsilon^k_{ij} := \delta_{N^k_{ij},1} \cdot \mathrm{incl}_j \ :\ \mathcal{P}_j \longrightarrow \bigoplus_{j'} \big(\mathcal{P}_{j'}\big)^{\oplus N^k_{ij'}} \tag{4.15}$$

for this "weighted" inclusion.

Proposition 4.7 Denote by

$$\Gamma^k_{ij;\,\alpha} := \mathrm{pr}_\alpha \circ \iota^k_i \circ \varepsilon^k_{ij} \;:\; \mathcal{P}_j \longrightarrow \mathcal{P}_k$$

the composition of the maps (4.13-15), and define the k-j-strings $C^k_{ij;\,\alpha}$ by

$$C^k_{ij;\,\alpha} := \sum_{|p\rangle\in\mathcal{P}_j} |\,\Gamma^k_{ij;\,\alpha}(p)\,\rangle\,\langle\,p\,|\ .$$

These $C^k_{ij;\,\alpha}$ satisfy the assumption of Proposition 4.4.

Proof: The proof is straightforward, using that $\Gamma^k_{ij;\,\alpha}$ maps elementary paths to elementary paths injectively, as well as the string multiplication rule:

$$\begin{aligned}\sum_{\alpha=1}^{n_i}\big(C^k_{im;\,\alpha}\big)^* C^k_{ij;\,\alpha} &= \sum_{\alpha=1}^{n_i}\sum_{|q\rangle\in\mathcal{P}_m}\sum_{|p\rangle\in\mathcal{P}_j} |\,q\,\rangle\,\langle\,\Gamma^k_{im;\,\alpha}(q)\,|\,\Gamma^k_{ij;\,\alpha}(p)\,\rangle\,\langle\,p\,| \\ &= \delta_{N^k_{ij},1}\sum_{|q\rangle\in\mathcal{P}_m}\sum_{|p\rangle\in\mathcal{P}_j} |\,q\,\rangle\,\delta_{p,q}\,\langle\,p\,| = \delta_{N^k_{ij},1}\,\delta_{m,j}\ \mathbf{1}_{\mathcal{P}_j} \qquad \blacksquare\end{aligned}$$

As an aside, let us mention the following applications of this proposition, or of eq. (4.10). The operator $C^k_{ij} \,:\, \mathcal{P}_j \longrightarrow \mathcal{P}_k^{\oplus n_i}$ given by

$$C^k_{ij} = \sum_{|p\rangle\in\mathcal{P}_j} |\,\iota^k_i(\varepsilon^k_{ij}(p))\,\rangle\,\langle\,p\,|\ , \tag{4.16}$$

which we can also write as a column vector $C^k_{ij} = \big(C^k_{ij;\,\alpha}\big)_{\alpha=1}^{n_i}$, is an *isometry*. Furthermore, the operators Π^k_{ij} and Π^k_i in $M_{n_i}(\mathcal{A}_k)$, defined as

$$\Pi^k_{ij} = C^k_{ij}\big(C^k_{ij}\big)^*\ , \qquad \Pi^k_i = \sum_{j\in I_q}\Pi^k_{ij}\ , \tag{4.17}$$

are both projections; in the first line, we regard $\big(C^k_{ij}\big)^*$ as a row vector of j-k-strings. When restricted to finite paths in $\mathcal{P}^{(n)}_k = \bigoplus_l \mathcal{P}^{(n)}_{k,l}$, cf. eq. (2.23), the rank of Π^k_i is $\sum_j N^k_{ij}\,\mathrm{dim}\mathcal{P}^{(n)}_j$. These operators will become important later when we will discuss the amplimorphisms of the observable algebra.

Clearly, the construction of the strings $C^k_{ij;\,\alpha}$ we have given is not the only way to solve (4.10). Nevertheless, we think that from the point of view of path spaces, our procedure is the most – and maybe even the only – natural one. Besides that, requiring that the embeddings ι^k_i are compatible with path prolongation reduces the amount of choices to be made quite drastically: ι^k_i is determined up to a unitary transformation in the finite subalgebra $M_{n_i}(\mathcal{A}^{(2)}_k)$ – with the further constraint that elementary paths should be mapped

to elementary paths. In the example below, it turns out that this essentially leaves only twists by certain permutation matrices in $M_{n_i}(\mathbb{C}\cdot\mathbf{1}_{\mathcal{A}_k})$.

All in all, our prescription how to construct charged field multiplets in the amplified path field algebra leads us to an almost unique and above all natural solution, which was possible by exploiting the "fine structure" of the underlying path spaces.

At the end of this section, let us again take a look at the case of the $c(2,5)$ minimal model. There, the non-trivial field multiplet F_1 is an element of $M_2(\mathcal{F})$, and according to Proposition 4.3, it has the following matrix structure

$$\big((F_1)_{\alpha\beta}\big)_{\beta=1}^2 = \begin{pmatrix} 0 & C^1_{11;\,\alpha} & 0 & 0 & 0 & 0 \\ C^1_{10;\,\alpha} & 0 & C^1_{11;\,\alpha} & 0 & 0 & 0 \\ 0 & 0 & 0 & C^1_{10;\,\alpha} & C^1_{11;\,\alpha} & 0 \end{pmatrix} \tag{4.18}$$

for $\alpha=1,2$. With the help of the explicit formula (4.6) for the amplimorphism ν_1 of $\mathfrak{g}_{(5)}$, it is straightforward to check covariance (4.7) of F_1 explicitly – and in this example, also uniqueness of the solution (4.18) is not too difficult to show.

The strings $C^k_{1j;\,\alpha}$ are constructed as above: For $k=0$, we have to choose embeddings $\iota^{0\ (n)}_{1,\,m} : \mathcal{P}^{(n)}_{1,m} \longrightarrow \mathcal{P}^{(n)}_{0,m}\oplus\mathcal{P}^{(n)}_{0,m}$ for both path ends $m=0,1$ and for $n\geq 2$ – or already for $n=1$ if possible: $\mathcal{P}^{(1)}_{0,0}=\emptyset$, but $\iota^{0\ (1)}_{1,\,1}$ can be defined: It maps the edge $(1\to 1)\in\mathcal{P}^{(1)}_{1,1}$ to one of the two copies of $(0\to 1)$ present in $\mathcal{P}^{(1)}_{0,1}\oplus\mathcal{P}^{(1)}_{0,1}$. This determines $\iota^{0\ (n)}_1$ on all paths in $\mathcal{P}^{(n)}_1$ that go through the node 1 on the first floor. In particular, one of the two copies of the path $(0;1,1)$ in $\mathcal{P}^{(2)}_{0,1}\oplus\mathcal{P}^{(2)}_{0,1}$ is in the image of those paths, so we have to map the remaining path $(1;0,1)\in\mathcal{P}^{(2)}_{1,1}$ to the other copy of $(0;1,1)$ in $\mathcal{P}^{(2)}_{0,1}\oplus\mathcal{P}^{(2)}_{0,1}$. This defines $\iota^{0\ (2)}_{1,\,m}$ and therefore ι^0_1 completely.

For $k=1$, we need to define $\iota^1_1 : \mathcal{P}_0\oplus\mathcal{P}_1 \longrightarrow \mathcal{P}_1\oplus\mathcal{P}_1$. The only natural possibility is to identify the space $\mathcal{P}_1$ on the left with one of the $\mathcal{P}_1$'s on the right, and then map $\mathcal{P}_0$ into the second copy of $\mathcal{P}_1$ with the "initial condition" $\iota^{1\ (1)}_{1,\,1}(0\to 1)=(1\to 1)\in\mathcal{P}^{(1)}_{1,1}$. Then, the only choice in the construction is which copy of $\mathcal{P}_1$ on the right to identify with the $\mathcal{P}_1$ on the left.

This means that both embeddings ι^k_1 are determined up to a permutation matrix in $M_2(\mathbb{C})$, acting trivially within the path representation spaces $\mathcal{P}_k$. Strictly speaking, however, we are not forced to map the space $\mathcal{P}_1$ in $\mathcal{P}_0\oplus\mathcal{P}_1$ "as a whole" into one of the $\mathcal{P}_1$ in the target space, but we could also "distribute" it over both copies. This relatively unnatural choice would introduce a higher degree of indeterminacy into our construction. Note, however, that in any case we can arrange ι^k_1 so as to map lowest energy states (the sequences of nodes $(0;1,1,1\ldots)$ or $(1;1,1,1\ldots)$) to lowest energy states again.

5. Global amplimorphisms

It is now very esay to write down global amplimorphisms of the path observable algebra which implement the charged sectors of our models. Copying the procedure of [62], we associate to each field multiplet $F_i \in M_{n_i}(\mathcal{F})$ the linear map

$$\rho_i \ : \ \begin{cases} \mathcal{A} \longrightarrow M_{n_i}(\mathcal{F}) \\ A \longmapsto \rho_i(A) = \Big(\rho_i(A)_{\alpha\beta}\Big)_{\alpha,\beta=1}^{n_i} \end{cases} \tag{5.1a}$$

with

$$\rho_i(A)_{\alpha\beta} := \sum_{\gamma=1}^{n_i} \big(F_i\big)_{\alpha\gamma}\, A\, \big(F_i^*\big)_{\gamma\beta}\ . \tag{5.1b}$$

Here and in the following, $\mathcal{A}$ is viewed as sub-algebra of $\mathcal{F}$ by the diagonal embedding as in the proof of Proposition 3.7. We list the relevant properties of these maps in a series of propositions. First we have to show that the maps ρ_i deserve the name *amplimorphisms of the global observable algebra*:

Proposition 5.1 The map ρ_i takes values in the n_i th amplification of the global path observable algebras $\mathcal{A}$. It is an injective *-homomorphism of AF-algebras.

Proof: By definition, $\rho_i(A)_{\alpha\beta}$ is an element of $\mathcal{F}$ for all $A \in \mathcal{A}$. In order to show that $\rho_i(A)_{\alpha\beta}$ are observables, it is sufficient to check that they commute with $U(a)$ for all $a \in \mathsf{g}$, see Proposition 3.7. Identifying a with $U(a)$ for simplicity, and using the summation convention, we compute for arbitrary $A \in \mathcal{A}$

$$\begin{aligned} a \cdot \rho_i(A)_{\alpha\beta} &= a(F_i)_{\alpha\gamma}\, A\, (F_i^*)_{\gamma\beta} = (F_i)_{\alpha\delta}(\nu_i(a))_{\delta\gamma}\, A\, (F_i^*)_{\gamma\beta} = (F_i)_{\alpha\delta}\, A\, (\nu_i(a))_{\delta\gamma}(F_i^*)_{\gamma\beta} \\ &= (F_i)_{\alpha\delta}\, A\, \big(F_i\nu_i(a)^*\big)^*_{\delta\beta} = (F_i)_{\alpha\delta}\, A\, \big(a^*F_i\big)^*_{\delta\beta} = (F_i)_{\alpha\delta}\, A\, (F_i^*)_{\delta\beta}\, a \\ &= \rho_i(A)_{\alpha\beta} \cdot a\ ; \end{aligned}$$

we have used Proposition 3.7 and the properties (4.7) and (4.8) of the covariant field multiplets repeatedly.
By construction, it is clear that ρ_i respects the *-operations of the path algebras, and multiplicativity is again established with the help of the covariance and completeness relations of the field multiplets:

$$\begin{aligned} \rho_i(A)_{\alpha\gamma}\rho_i(B)_{\gamma\beta} &= (F_i)_{\alpha\delta}\, A\, (F_i^*)_{\delta\gamma}\, (F_i)_{\gamma\epsilon}\, B\, (F_i^*)_{\epsilon\beta} \\ &= \big(F_i \cdot \nu_i(1)\big)_{\alpha\epsilon}\, AB\, (F_i^*)_{\epsilon\beta} = \big(\rho_i(AB)\big)_{\alpha\beta} \end{aligned}$$

for all elements A, B of the path observable algebra $\mathcal{A} \subset \mathcal{F}$.
That ρ_i is an injective map from $\mathcal{A}$ to $M_{n_i}(\mathcal{A})$ also follows directly from (4.8): "Sandwiching" $\rho_i(A)$ by F_i^* and F_i, and using the multiplicativity of the g-amplimorphism ν_i, we obtain

$$\big(F_i^*\, \rho_i(A)\, F_i\big)_{\alpha\beta} = \nu_i(1)_{\alpha\gamma}\, A\, \nu_i(1)_{\gamma\beta} = A\, \nu_i(1)_{\alpha\beta}$$

for all $\alpha, \beta = 1, \ldots, n_i$. Thus, $\rho_i(A) = 0$ implies $A = 0$. ∎

Note that these properties do not hold if we extend ρ_i to all of $\mathcal{F}$ – and also that the proof did not depend on any specific properties tied to the path representations. Those came into play when we had to solve the conditions (4.7) and (4.8) for the field multiplets explicitly by constructing the strings $C^k_{ij;\,\alpha}$, which appear in the concrete expressions for the amplimorphisms.

In order to obtain such formulas, we use the decomposition $A = A_0 + \ldots + A_K$ of elements of $\mathcal{A} = \mathcal{A}_1 \oplus \ldots \oplus \mathcal{A}_K$, and the diagonal embedding (3.15) of $\mathcal{A}$ into the field algebra. Inserting the covariant field multiplets from Proposition 4.3 into (5.1), and keeping track of the Kronecker symbols, yields the following expressions for the elements of the amplimorphisms:

$$\big(\rho_i(A)\big)_{(\alpha;n_k,\sigma)(\beta;n_j,\tau)} = \delta_{k,j}\,\delta_{\sigma,\tau} \sum_{l\in I_q} \delta_{N^k_{il},1}\, C^k_{il;\,\alpha}\, A_l\, \big(C^k_{il;\,\beta}\big)^* \tag{5.2}$$

with $C^k_{il;\,\alpha}$ as in Proposition 4.7; the matrix notation is as in section 4.1. By construction of the string coefficients, ρ_i maps the n-floor finite-dimensional sub-algebra $\mathcal{A}^{(n)}$ of $\mathcal{A}$ into $M_{n_i}(\mathcal{A}^{(n)})$ and is compatible with the embeddings $\Phi^{(n)}$ induced by path concatenation. We say that the amplimorphisms *preserve the filtration* of the systems $\big(\mathcal{A}_i^{(n)}, \Phi^{(n)}\big)$.
In addition, formula (5.2) makes it clear that the amplimorphisms ρ_i are *non-unital* in general. We have

$$\rho_i(\mathbf{1})_{(\alpha;n_k,\sigma)(\beta;n_j,\tau)} = \delta_{k,j}\,\delta_{\sigma,\tau}\,\big(\Pi^k_i\big)_{\alpha\beta}$$

with the projection $\Pi^k_i \in M_{n_i}(\mathcal{A}_k)$ from eq. (4.17), which is $\neq \mathbf{1}$ for $i \neq 0$.

During the construction of the field multiplets, we have made several choices, and we must determine to which extend the amplimorphisms ρ_i depend on them:

Proposition 5.2 The $\mathcal{A}$-amplimorphisms ρ_i depend only on the isomorphism class of the g-amplimorphisms ν_i which enter the covariance law (4.7) of the field multiplets. For a given class of ν_i, ρ_i are unique up to conjugation by unitaries in $M_{n_i}(\mathcal{A})$.

Proof: The second statement follows immediately from the $C^k_{ij;\,\alpha}$ dependence of the $\mathcal{A}$-amplimorphisms. To see the first claim, assume that we "twist" the g-amplimorphism ν_i by a unitary $u \in M_{n_i}(\mathsf{g})$ and use $\nu'_i := \mathrm{Ad}_u \circ \nu_i$ instead. Then the covariant field multiplets change as $F'_i := F_i \cdot u^{-1}$, where u^{-1} acts through the representation U of g on $\mathcal{H}^{\mathrm{tot}}$, see Definition 3.6. Since $U(u)$ commutes with all elements of $\mathcal{A}$, formula (5.1) for ρ_i stays invariant under a twist. ∎

The isomorphism classes of the g-amplimorphisms were of course fixed by the $c(2,q)$ fusion rules from the start, therefore we conclude that our construction yields filtration-preserving amplimorphisms of the global path observable algebra of unique isomorphism type. But moreover, in view of the remarks on the choice of strings $C^k_{ij;\,\alpha}$ made after Proposition 4.7, the ρ_i which are "natural" from the path perspective are fixed up to conjugation by a permutation matrix in $M_{n_i}(\mathcal{A}^{(2)})$.

We can now ask whether the ρ_i implement the representations of the global observable algebra on the vacuum sector:

Proposition 5.3 Let $\pi_i \,:\, \mathcal{A} \longrightarrow \mathcal{A}_i$ be the projection of $\mathcal{A}$ onto the representation in the sector $\mathcal{H}_i$, and denote the extensions to amplifications $M_n(\mathcal{A})$ by the same symbol. Then the amplimorphisms ρ_i of eq. (5.1) satisfy

$$\pi_0 \circ \rho_i \simeq \pi_i \ ,$$

where $\simeq$ denotes equivalence by isometries.

PROOF: The statement follows immediately from formula (5.2). Setting $k = 0$, the condition $N_{il}^0 = 1$ enforces $l = i$ (uniqueness of the conjugated sector), and we remain with

$$\pi_0\big(\rho_i(A)\big)_{\alpha\beta} = C^0_{ii;\,\alpha}\, A_i\, \big(C^0_{ii;\,\beta}\big)^* \ .$$

Thus, the string isometries C^0_{ii} from eq. (4.16) 'transport" the representation $\mathcal{A}_i$ into $M_{n_i}(\mathcal{A}_0)$ and implement the equivalence. Note that by our construction of the $C^k_{ij;\,\alpha}$ the equivalence also holds when $\mathcal{A}$ and $\mathcal{A}_i$ are replaced by finite-dimensional sub-algebras $\mathcal{A}^{(n)}$ and $\mathcal{A}_i^{(n)}$. ∎

Proposition 5.4 Upon composition, the amplimorphisms realize the fusion rules of the $c(2,q)$ minimal models, i.e. the relations

$$\pi_0 \circ\, (\rho_i \circ \rho_j) \ \ \simeq \ \ \bigoplus_{k\in I_q} \bigoplus_{r=1}^{N_{ij}^k} \pi_0 \circ \rho_k$$

hold in $M_{n_in_j}(\mathcal{A}_0)$.

PROOF: We apply the lhs to $A \in \mathcal{A}$ and find after a short calculation

$$\pi_0\big(\rho_i(\rho_j(A))\big)_{\alpha\alpha',\beta\beta'} = \sum_{k\in I_q} \delta_{N^k_{ij},1}\, \big(C^0_{ii;\,\alpha}\, C^i_{jk;\,\alpha'}\big)\, A_k\, \big(C^0_{ii;\,\beta}\, C^i_{jk;\,\beta'}\big)^*$$

where $\alpha, \beta = 1, \ldots, n_i$ and $\alpha', \beta' = 1, \ldots, n_j$. Again, the claim follows from the completeness and orthogonality relation (4.10) for the strings $C^k_{ij;\,\alpha}$. ∎

In view of all these properties, we may regard the amplimorphisms ρ_i as variants of the DHR morphisms of algebraic quantum field theory.

6. Open problems

Having constructed amplimorphisms for the $c(2,q)$ models, one may of course ask whether they yield further information on these conformal field theories. Since the algebraic approach is superior to all others when it comes to the discussion of braid group statistics, one should in particular try to compute intertwiners (statistics operators), left inverses and Markov traces associated to the morphisms of the observable algebra. In this way, interesting braid group representations and knot invariants might arise and, as a by-product, one could supply the QSA g with the full data of a weak quasi-triangular quasi Hopf algebra. The results of [62] indicate that dealing with non-unital amplimorphisms rather that unital endomorphisms does not pose severe problems, and we hope that this program can be carried out even in the absence of local information in our path algebras and morphisms.

Nevertheless, if we aim at a complete description of the $c(2,q)$ models within the algebraic framework, it is important to recover the local net structure inside the global algebras, and to show that our amplimorphisms are equivalent to localizable ones. One possible starting point for the construction of local sub-algebras of the path algebra is provided by the su(1,1) action on the path spaces constructed in [56]. But although there seems to be no principle problem, it is technically rather difficult to implement the conditions of Möbius covariance on the local sub-algebras.
In this context, we conjecture that it is precisely because they are filtration-preserving that our global amplimorphisms have good chances to be equivalent to covariant and localized morphisms: The su(1,1)-action of [56] is designed in analogy with the excitation of quasi-particles and therefore respects the finite length filtration of the path spaces as much as possible.
Given an action of su(1,1) or even the Virasoro algebra on the path spaces, we could also make our somewhat abstract amplimorphisms more concrete: Although their action on path algebra elements can be made as explicit as we wish, it would be interesting to have formulas for ρ_i applied to Virasoro generators, similar to [44] where neat expressions could be obtained because the Virasoro modes have simple expansions in terms of the fermion modes, which in turn are acted on by the endomorphisms in a simple fashion.

In addition, once sub-algebras of local observable have been identified within the global path algebra, one would also like to make contact to the von Neumann algebra description of local QFTs with all its particular merits. It remains to be seen whether one then meets problems with the non-unitarity of the $c(2,q)$ models, which apparently played no role at the purely algebraic level of our global considerations.

We have seen in this paper that the path representations of the $c(2,q)$ minimal models open up a lot of interesting possibilities. They allow to organize a great deal of information on these CFTs into a single labeled graph; they naturally lead to an AF-algebraic description of theories which a priori are defined in terms of unbounded Virasoro modes; they can, in this respect, be regarded as an alternative to the usual free field constructions; they seem to encode, via the quasi-particle reformulation, structural details of non-conformal relatives of the $c(2,q)$ minimal models within the CFT and might, therefore, even be useful in the context of massive integrable quantum field theories.

In view of these facts, it is desirable to find similar path representations for other conformal models as well. In [41], this has been achieved for a subset of modules in the $c(\text{even},\text{odd})$ Virasoro minimal models, with our graphs $\mathcal{G}_q$ again playing an important role. However, since some of the sectors do not have a path description, our construction cannot be applied to those models, yet.

Whereas the results in [41] have been obtained by factorization of some of the characters of the minimal models, i.e. by purely combinatorial means, one could alternatively try to imitate the FNO procedure in order to determine explicit bases of the irreducible modules. However, for general minimal models the structure of the annihilating ideal is more involved than in the $c(2,q)$ cases, and it seems necessary to pass to the maximally extended chiral observable algebra before progress can be made.

We also came across some problems of a more abstract nature: In the introduction, we raised the speculation that there is a general relation between AF-algebras and conformal field theories. Somewhat related to this conjecture, we have seen that the global path observable algebras of the $c(2,q)$ models are of the same type as the string algebras which show up as intertwiner (symmetry) algebras in the DHR framework. Thus it seems that in the case of these conformal models, internal and space-time symmetries are indeed "inexorably linked" [59]. Still, the relationship remains to be made precise. On the other hand, we have shown that there also is a canonical semi-simple QSA for any quasi-rational CFT. The question is whether one can find an axiomatic foundation of the notions introduced in [48]. Since the dimensions of the small spaces provide new invariants of quasi-rational CFTs, this might lead to interesting developments within algebraic QFT and even in the theory of operator algebras.

Acknowledgements

I would like to thank A. Alekseev, D. Buchholz, K. Fredenhagen, J. Fröhlich, J. Fuchs, A. Ganchev, K. Gawędzki, W. Nahm, V. Schomerus, B. Schroer, K. Szlachanyi and P. Vecsernyes for useful discussions, comments and encouragement. I am particularly indebted to M. Rösgen whose collaboration on path representations was invaluable.
This work was partially support by a HCM Fellowship of the European Union.

Figure 1: Fusion graphs $\mathcal{G}_q$ of the minimal dimension field for $q = 5,\ 7$ and 9; the labels i at the nodes refer to the sectors ϕ_i :

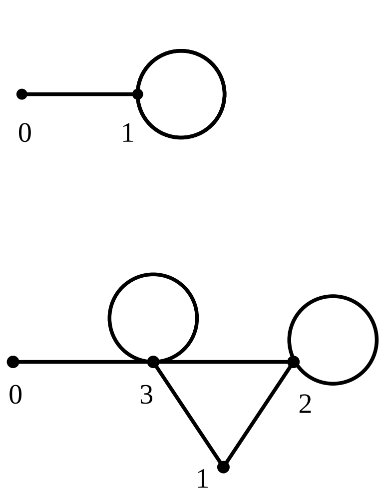
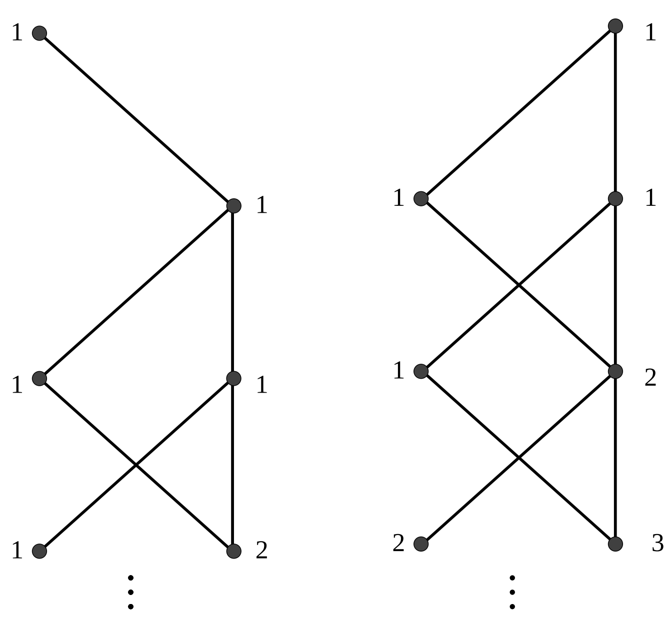
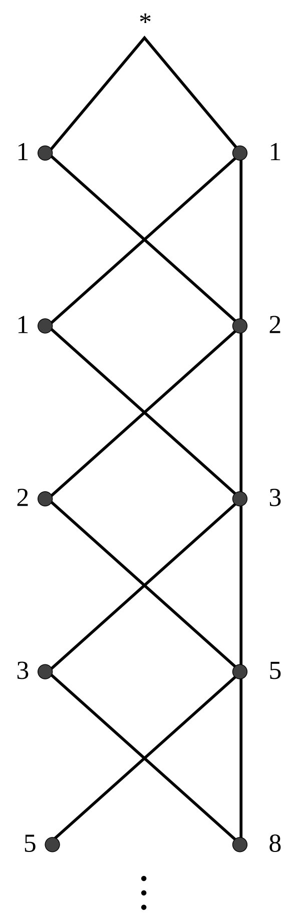


Figure 2: Bratteli diagrams $\mathcal{B}_{5,0}$, $\mathcal{B}_{5,1}$ and $\widehat{\mathcal{B}}_5$, associated to the observable algebras in the vacuum sector, in the $h = -\frac{1}{5}$ representation, and to the path field algebra of the $c(2,5)$ model. Labels on the n th floor give the dimensions of the spaces of paths running in n steps from the top node (on the zeroth or -2 nd floor of the diagram) to the node 0 (on the left) or 1 (on the right) of $\mathcal{G}_5$:





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